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EMBEDDED MINIMAL SURFACES IN 3-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

J. H. RUBINSTEIN

ABSTRACT. Let M be a closed orientable Riemannian 3-manifold with positive scalar curvature. We prove that any embedded closed minimal surface in M has a topological description as a generalized Heegaard surface. Also an existence theorem is proved which gives examples of such minimal surfaces.

0. Let M^3 be a closed orientable 3-manifold equipped with a Riemannian metric with positive scalar curvature. A closed surface $L \subset M$ is called *minimal* if the mean curvature of L is zero everywhere. This is equivalent to the condition that, for all variations L_t of L, the area A_t of L_t is stationary at L, i.e., $A'_0 = 0$.

Our aim is to give a simple topological description of such surfaces L and to establish an existence result yielding some examples to illustrate the various cases that can arise. We now give a brief survey of previous work in this area.

We call $L \subset M$ a *Heegaard surface* if $M - L$ has 2 components whose closures are handlebodies. Heegaard surfaces $L, L' \subset M$ are said to be *equivalent* if there is a diffeomorphism $\phi: M \to M$ with $\phi(L) = L'$. Lawson **[11]** showed that if M is a Riemannian S^3 with positive Ricci curvature then any embedded minimal surface is a Heegaard surface. Note also that Waldhausen **[19]** proved that any two Heegaard surfaces of the same genus in S^3 are equivalent. Lawson **[10]** gave examples of embedded minimal surfaces of every genus in S^3 with the standard metric. In addition, many such examples are constructed in **[10]** for other 3-manifolds with constant curvature 1, i.e. of the form S^3/Γ where $\Gamma \subset SO(4)$ acts freely on S^3. Amongst these are pairs of embedded minimal surfaces in S^3 with the same genus for which there is no isometry of S^3 taking one onto the other.

More recently, Meeks-Simon-Yau (see §8 of **[12]**) established the same result as Lawson **[11]** for any M^3 with positive Ricci curvature. Also they obtained a topological characterization of orientable closed minimal surfaces embedded in $\#_{i=1}^n S^2 \times S^1$ equipped with a Riemannian metric with nonnegative scalar curvature (see §10 of **[12]**). We will give similar results for any M with a metric with positive scalar curvature (or nonnegative scalar curvature if M does not admit a flat metric) and will also treat nonorientable minimal surfaces.

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1. In this section, we collect together some related results and definitions.

THEOREM 1 [**6**, **18**]. *Let M be a closed orientable 3-manifold which admits a Riemannian metric with positive scalar curvature. Then M can be written as a connected sum $M_1 \# \cdots \# M_k$, where for each i either M_i is a copy of $S^2 \times S^1$ or $\pi_1(M_i)$ is finite.*

REMARK. It is conjectured that every closed 3-manifold with finite fundamental group is diffeomorphic to a standard spherical 3-manifold with constant curvature 1. This has been proved by Hamilton [**8**] if the 3-manifold has a metric with positive Ricci curvature.

THEOREM 2 [**5**, **16**]. *Any 3-manifold M which is a finite connected sum of copies of $S^2 \times S^1$ and of standard spherical 3-manifolds admits a Riemannian metric with positive scalar curvature.*

We now state a result which is proved in [**17**] for closed 3-manifolds. The same argument clearly works in the case of compact 3-manifolds with suitable boundary.

THEOREM 3 [**17**]. *Let M be a compact 3-manifold with a Riemannian metric with positive scalar curvature and assume that ∂M has nonnegative mean curvature with respect to the outward normal. Then $\pi_1(M)$ has no subgroups of the form $\pi_1(J)$, where J is a closed orientable surface with genus > 0.*

DEFINITION (CF. [**14**]). A closed nonorientable surface K embedded in a closed orientable 3-manifold M is called a *one-sided Heegaard surface* if $M - K$ is an open handlebody.

REMARK. If $K \subset M$ is a one-sided Heegaard surface then M has a double covering $\tilde{M}$ so that the preimage of K in $\tilde{M}$ is a Heegaard surface $\tilde{K} \subset \tilde{M}$ (cf. [**14**] and §2).

DEFINITION. (1) Suppose $L^2 \subset M^3$ where L, M are closed and orientable. L is called a *partial Heegaard surface* for M if one of the following holds:

(a) L separates M into 2 regions, each of which is a connected sum of an open handlebody with some closed 3-manifold.

(b) $M - L$ is the connected sum of 2 open handlebodies with a closed 3-manifold.

(2) If K is a closed nonorientable surface then $K \subset M$ is a *one-sided partial Heegaard surface* in M if $M - K$ is the connected sum of an open handlebody with a closed 3-manifold.

REMARKS. By Kneser [**9**] and Milnor [**13**], any closed orientable M^3 can be uniquely expressed as a finite connected sum of prime 3-manifolds. (A closed orientable 3-manifold Q is prime if either $Q = S^2 \times S^1$ or any embedded S^2 in Q bounds a 3-cell.) Hence, if L is a separating partial Heegaard surface, then M has a prime factorization $M = M_1 \# \cdots \# M_{k+r+s}$ where L can be viewed as a Heegaard surface for $M_1 \# \cdots \# M_k$, $M_{k+1} \# \cdots \# M_{k+r}$ lies on one side of L and $M_{k+r+1} \# \cdots \# M_{k+r+s}$ is on the other side. Similarly, if L is nonseparating, M has a prime decomposition $M = M_1 \# \cdots \# M_{k+r} \# S^2 \times S^1$ where L is a Heegaard surface for $M_1 \# \cdots \# M_k$ and the $S^2 \times S^1$ factor arises by forming a connected sum of the two open handlebodies which are the components of $M_1 \# \cdots \# M_k -$

L. The other factors $M_{k+1}, \ldots, M_{k+r}$ all miss L. Finally if K is a one-sided partial Heegaard surface in M, then $M = M_1 \# \cdots \# M_{k+r}$ where each M_i is prime, K is a one-sided Heegaard surface for $M_1 \# \cdots \# M_k$ and the other factors M_{k+i}, for $1 \leqslant i \leqslant r$, are all disjoint from K.

DEFINITION. A closed surface $L \subset M$ of genus > 0 is called *incompressible* if there is no disk $D \subset M$ with $D \cap L = \partial D$ a noncontractible loop on L. A 2-sphere $S \subset M$ is incompressible if S does not bound a 3-cell in M.

2. Suppose K is a closed nonorientable surface embedded in a closed orientable 3-manifold M. We will construct various double coverings $\tilde{M}$ of M for which the preimage $\tilde{K}$ of K in $\tilde{M}$ is orientable. Let $M = N(K) \cup Y$, where $N(K)$ is a small closed regular neighbourhood of K and $Y = M - \operatorname{int} N(K)$. Let $\tilde{K}$ be the orientable double covering of K and let $\tilde{Y}$ be any double covering of Y such that $\partial\tilde{Y}$ is disconnected. We allow the possibility that $\tilde{Y}$ itself is disconnected and, in this case, $\tilde{Y}$ is the disjoint union of two copies of Y. Note that a connected double covering $\tilde{Y}$ of Y exists if and only if the map $H_1(K, Z_2) \to H_1(M, Z_2)$ induced by inclusion is not onto. There is a double covering of $N(K)$ by $\tilde{K} \times [-1,1]$ with the preimage of K equal to $\tilde{K} \times \{0\}$, since $N(K)$ is a twisted line bundle over K.

The double covering $\tilde{M}$ is obtained by gluing $\tilde{K} \times [-1,1]$ to $\tilde{Y}$ by a diffeomorphism ψ: $\tilde{K} \times \{-1,1\} \to \partial\tilde{Y}$. ψ is chosen so that the double covering projections $\tilde{K} \times [-1,1] \to N(K)$ and $\tilde{Y} \to Y$ match up. So we obtain $\tilde{M} = \tilde{K} \times [-1,1] \cup_\psi \tilde{Y}$ and there is a double covering p: $\tilde{M} \to M$ with $p^{-1}(K) = \tilde{K} \times \{0\}$.

3. THEOREM 4. *Suppose M is a closed orientable Riemannian 3-manifold. Assume that the scalar curvature is positive, or if M admits no flat metric then it suffices to suppose that the scalar curvature is nonnegative. If $L \subset M$ is minimal then L is a partial Heegaard surface for M.*

PROOF. This will follow by applying Theorem 3 to M split along L. Firstly, if L is orientable and nonseparating, we obtain a connected compact manifold M_1 with two copies of L in ∂M_1, by dividing M along L. If L separates M, two compact manifolds M_2 and M_3 are constructed with ∂M_2 and ∂M_3 both a copy of L, by splitting M along L. Finally, if L is nonorientable, let p: $\tilde{M} \to M$ be the double covering constructed in §2, where $\tilde{M} = \tilde{L} \times [-1,1] \cup_\psi \tilde{Y}$ and $\tilde{Y}$ is disconnected. Let Y_1, Y_2 be the components of $\tilde{Y}$ with $\partial Y_1 = \psi(\tilde{L} \times \{-1\})$. The compact manifold $M_4 \subset \tilde{M}$ given by $M_4 = \tilde{L} \times [-1,0] \cup Y_1$ satisfies ∂M_4, is a copy of $\tilde{L}$ and p maps int M_4 diffeomorphically onto $M - L$.

By Theorem 3, there are no subgroups of $\pi_1(M_i)$ of the form $\pi_1(J)$, where J is a closed orientable surface of positive genus and $1 \leqslant i \leqslant 4$. Note that if M is assumed only to have a metric of nonnegative scalar curvature, then as in **[17]** the metric can be approximated by one with positive scalar curvature (since M admits no flat metric) and Theorem 3 applies.

To complete the proof we apply Dehn's lemma and the Loop theorem to the surfaces ∂M_i, for $1 \leqslant i \leqslant 4$. Suppose G is a component of ∂M_i and $\{D_u: 1 \leqslant u \leqslant v\}$ is a maximal family of disjoint compressing disks for G, i.e. $\partial D_u \subset G$, int $D_u \subset$ int M_i,

∂D_u is noncontractible in G for all u and no two curves ∂D_u and ∂D_w are parallel on G, for $u \neq w$. Let $N = N(\bigcup_u D_u \cup G)$ be a small closed regular neighbourhood in M_i. If some component J of $\partial N - G$ has genus > 0, then since $\pi_1(J) \subset \pi_1(M_i)$ (by Dehn's lemma and the Loop theorem) we get a contradiction to Theorem 3. So ∂N consists of G together with 2-spheres. It follows immediately that M_2, M_3 and M_4 are all connected sums of a handlebody (N with cells attached along the 2-spheres in ∂N) with a closed 3-manifold. Similarly, M_1 is a connected sum of two handlebodies with a closed 3-manifold, and so L is a partial Heegaard surface in all cases.

REMARKS. (1) In the case that $M^3 = \#_{i=1}^n S^2 \times S^1$, Theorem 4 gives the results in §10 of [**12**]. By results of Haken [**7**] and Waldhausen [**19**], any two orientable partial Heegaard surfaces of the same genus in $\#_{i=1}^n S^2 \times S^1$ are equivalent if they are both separating or both nonseparating and if there are the same numbers of $S^2 \times S^1$ factors in the components of the complements of the surfaces (cf. [**12**, §10] also).

(2) Engmann [**4**] and Birman [**1**] have given examples of two Heegaard surfaces of genus 2 in a connected sum of lens spaces $L(p, q) \# L(p', q')$ which are not equivalent. On the other hand, Bonahon and Otal [**3**] have recently shown that any two Heegaard surfaces with the same genus in any lens space $L(p, q)$ are equivalent.

4. THEOREM 5. *Suppose M is a closed orientable Riemannian 3-manifold with no orientable incompressible surfaces of genus > 0. If $H_1(M, Z)$ has 2-torsion then M has an embedded minimal nonorientable surface which is a one-sided partial Heegaard surface. Consequently, there are double covers $\tilde{M}$ of M with orientable minimal partial Heegaard surfaces.*

PROOF. Since $H_1(M, Z)$ has 2-torsion, we can choose a torsion element $\alpha \in H_1(M, Z)$ such that $\alpha \otimes 1 \neq 0$ in $H_1(M, Z) \otimes Z_2$. Exactly as in [**14**], a nonorientable incompressible surface $K \subset M$ can be found so that the intersection number mod 2 of the class of K in $H_2(M, Z_2)$ and α is one. Moreover, K is a one-sided partial Heegaard surface for M, because there are no orientable incompressible surfaces in M other than 2-spheres. (In [**14**] it is proved that K is a one-sided Heegaard surface if M is irreducible. Clearly the same argument works here.) By [**12**] there is a stable minimal incompressible surface $K' \subset M$, with K' homeomorphic to K. K' is also a partial Heegaard surface in M. The preimage $\tilde{K}'$ of K' in a double cover $\tilde{M}$ of M, as constructed in §2, is then an orientable minimal partial Heegaard surface for $\tilde{M}$. This completes the proof.

EXAMPLES. (1) The above hypotheses are satisfied by a finite connected sum $M = M_1 \# \cdots \# M_k$, where each M_i is either a copy of $S^2 \times S^1$ or $\pi_1(M_i)$ is finite and at least one $\pi_1(M_i)$ is even, solvable and not generalised binary tetrahedral by cyclic.

(2) Let $M = L(4, 1) \# S^2 \times S^1$ with any Riemannian metric. Now $L(4, 1)$ has an incompressible Klein bottle K (see [**14**]) which can be minimally embedded in M, as in Theorem 5. M has two double coverings, $\tilde{M}_1 = RP^3 \# S^2 \times S^1$ and $\tilde{M}_2 = RP^3 \# S^2 \times S^1 \# S^2 \times S^1$, in which K is covered by a minimal torus $\tilde{K}$ (cf. §2). In the former, the torus is nonseparating, while in the latter case, it is a separating surface. So we obtain all three types of partial Heegaard surfaces realized by minimal surfaces.

(3) All the lens spaces $L(2p, q)$ have embedded incompressible nonorientable surfaces K (cf. [**2** and **14**] for properties of these surfaces). As in Theorem 5, K can be chosen to be minimal and lifts to minimal Heegaard surfaces in both $L(p, q)$ and S^3. For example, with the standard spherical metric on $L(2p, q)$ this produces many new embedded minimal surfaces in S^3, as compared with Lawson's constructions in [**10**]. See also [**15**] for a related procedure for finding minimal surfaces, based on an equivariant version of [**12**].

NOTE ADDED IN PROOF. Similar results have been obtained by S. Almeida, Ph. D. Thesis, State University of New York, Stony Brook (December 1982).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VICTORIA 3052, AUSTRALIA