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FOLD SINGULARITIES IN PSEUDO RIEMANNIAN GEODESIC TUBES

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ABSTRACT. For a general submanifold of a pseudo Riemannian manifold, the exponential mapping of the orthogonal bundle into the ambient manifold may fail to be a diffeomorphism on the zero section. Here we show that differential geometric information intrinsic to the submanifold determines when this map has a fold singularity.

Throughout this paper $i\colon N \to (M, \langle\ \rangle)$ will denote a C^∞ immersion of smooth manifolds of dimension n and m, respectively. M will come equipped with a smooth nondegenerate geodesically complete pseudo Riemannian metric. If $N^\perp$ denotes the orthogonal bundle over N, $N_p^\perp = \{v_q \in T_qM \mid i(p) = q,\ v_q \perp i_*(T_pN)\}$, then the geodesic tube map $\bar{i}\colon N^\perp \to M$ is defined by $v_p \to \exp_{i(p)}v$. If $(M, \langle\ \rangle)$ is Riemannian (i.e. $\langle\ \rangle$ is positive definite), then $\bar{i}$ is a local diffeomorphism on a neighborhood of any point on the zero section of $N^\perp$. If $(M, \langle\ \rangle)$ is not definite this is not always the case. In particular, if $N_p^\perp \cap T_pN$ is a nontrivial vector space then it is clear that $(\bar{i})_*$ cannot have full rank at $o_p \in N^\perp$. The purpose of this paper is to show that in the case where $N_p^\perp \cap T_pN$ is one-dimensional the intrinsic geometry of N (i.e. invariants associated with $(N, i^*\langle\ \rangle)$) determine when the geodesic tube map has a simple fold singularity at o_p (i.e. a $S_{1,0}$ singularity; see Golubitsky and Guillemin, p. 87). The author encountered this phenomena in a geometric study of a special class of first order partial differential equations. There it is used to determine certain qualitative features of solutions.

One may pose the problem of describing how the geometry of $i\colon N \to (M, \langle\ \rangle)$ determines the structure of more complicated singularities in the geodesic tube map. Aside from shedding light upon some of the finer features of pseudo Riemannian geometry, the resolution of this matter would be of use in the study of the aforementioned first order partial differential equations. The author wishes to express thanks to R. Bryant, J. Damon, P. Eberlein, R. B. Gardner, M. Schlessinger and G. Thompson for inspiration.

Our program is as follows. Because $N_p^\perp \cap T_pN \neq 0$ implies that $i^*\langle\ \rangle_p$ is a degenerate bilinear form we begin by adapting the classical connection construction to such a setting. This is the dual connection. At a point p where $i^*\langle\ \rangle$ is degenerate the dual connection is used to define a conformal structure on T_pN. We show that this conformal structure determines when the degeneracy of $i^*\langle\ \rangle$ satisfies two

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transversality conditions. We then show that if these conditions are satisfied at p then the geodesic tube map has a fold singularity at o_p. (The reader may wish to consult the addendum to review the notion of fold singularity.) We close with a simple example.

Given a smooth manifold N with $p \in N$ let $C^\infty(N)$, $V^\infty(N)$, $F^r(N)$ denote the collections of smooth functions, vector fields and r-forms on N. The subscript p will denote the germs of such objects at p. $\mathscr{M}_p(N) \subset C_p^\infty(N)$ will denote the maximal ideal of germs of functions which vanish at p, and $\odot^2 T^*N$ will denote the symmetric $(0,2)$ tensor bundle over N. We have dscrm: $\odot^2 T^*N \to \mathbf{R} \bmod \mathbf{R}^*$ which assigns to a point in a fiber its discriminant. (Recall the discriminant of a symmetric bilinear form vanishes iff the bilinear form degenerates.) Keeping $(N, i^*\langle\ \rangle)$ in mind we define a *manifold with metric* to be a pair $(N, \langle\ \rangle)$ where $\langle\ \rangle$ is *any* C^∞ section of $\odot^2 T^*N$. $(N, \langle\ \rangle)$ is *singular* at $p \in N$ if $\operatorname{dscrm} \circ \langle\ \rangle_p \equiv 0$. It is known that $\operatorname{dscrm}^{-1}(0) \subset \odot^2 T^*N$ is smoothly stratified according to type (r, s) where $r + s < n$. Recall that a symmetric bilinear form is of type (r, s) if it is of rank $r + s$, and admits positive and negative definite subspaces of dimensions r and s, respectively. If $\langle\ \rangle: N \to \odot^2 T^*N$ has nonempty transverse intersection with the stratum of type (r, s), $r + s < n$, at $p \in N$ we will say $(N, \langle\ \rangle)$ has a *transverse metric singularity* at p and write $(N, \langle\ \rangle) \pitchfork_p S(r, s)$. The *codimension* of the singularity is the codimension of the (r, s) stratum in $\odot^2 T^*N$. In this paper we will consider only metric singularities of codimension one. If such a singularity is transverse the locus of points S at which the metric is singular is a smooth submanifold of dimension $n - 1$. The following serves as a local characterization of such metric singularities.

PROPOSITION 1. *Given $(N, \langle\ \rangle)$ with $\langle\ \rangle_p$ of type (r, s), $r + s = n - 1$, then $(N, \langle\ \rangle) \pitchfork_p S(r, s)$ iff, for some C^∞ frame $e_1, \ldots, e_n \in V_p^\infty(N)$, $d(\det\langle e_i, e_j\rangle)_p \neq 0$.*

PROOF. Omitted.

Given such a metric singularity we have two subspaces of T_pN, namely the tangent space to the manifold of singular points T_pS and the one-dimensional subspace Rad_p which is orthogonal to all of T_pN. If these subspaces split T_pN we say the metric singularity is *radical transverse* at p. Let us see how these transversality conditions are geometrically encoded in $(N, \langle\ \rangle)$.

Now let $(N, \langle\ \rangle)$ be a smooth manifold with $\langle\ \rangle$ *any* smooth symmetric $(0,2)$ tensor.

DEFINITION 1. Given $(N, \langle\ \rangle)$, a C^∞ *dual connection* on N is a map

$$\square: V^\infty(N) \times V^\infty(N) \to F^1(N),$$
$$X, Y \to \square_X Y,$$

which satisfies

(a) $\square$ is $\mathbf{R}$ bilinear,

(b) for all $f \in C^\infty(N)$ and $X, Y \in V^\infty(N)$,

(1) $\square_{fX} Y = f \square_X Y$,

(2) $\square_X fY = X(f)\langle Y, \ \rangle + f \square_X Y$,

the *torsion* of $\square$ is the $(0,3)$ tensor

$$\operatorname{Tor}(X, Y, Z) = \square_X Y(Z) - \square_Y X(Z) - \langle [X, Y], Z\rangle;$$

here $\square_X Y(Z)$ denotes the pairing of the 1-form $\square_X Y$ with the vector field Z. $\square$ is compatible with $\langle\ \rangle$ if

$$X\langle Y, Z\rangle = \square_X Y(Z) + \square_X Z(Y).$$

LEMMA 2. *Given $(N, \langle\ \rangle)$ there exists a unique torsion free dual connection which is compatible.*

PROOF. As in the classical case we use the relations of compatibility and $\operatorname{Tor} = 0$ to write

$$(*)\qquad \begin{aligned} 2\square_X Y(Z) = {} & X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned}$$

One then shows that this object satisfies (a) and (b).

Observe that if $\langle\ \rangle$ is nondegenerate then $\square_X Y(Z) = \langle \nabla_X Y, Z\rangle$, where ∇ is the classical Levi-Civita connection.

Now let $R_p \in \operatorname{Rad}_p$ and X_p, $Y_p \in T_pN$ and set $\mathrm{II}_p(X, Y, R) = \square_{X_p} Y(R)$.

PROPOSITION 3. (a) *II_p is a tensor on $T_pN \times T_pN \times \operatorname{Rad}_p$*,
(b) $\mathrm{II}_p(X, Y, R) = \mathrm{II}_p(Y, X, R)$.

PROOF. (a) Observe $\square_{X_p} fY(R) = X(f)\langle Y, R\rangle_p + f\square_{X_p} Y(R)$ and $\langle Y, R\rangle_p = 0$.
(b) Observe $0 = \langle R, [X, Y]\rangle_p = \square_{X_p} Y(R) - \square_{Y_p} X(R)$.

Because there is no natural choice of $R_p \subset \operatorname{Rad}_p$ we view $\mathrm{II}_p(\ ,\ , R)$ as a conformal structure on T_pN, the *intrinsic conformal structure* of a metric singularity. The two transversality conditions on a metric singularity are encoded in II_p as follows.

PROPOSITION 4. *Given $(N, \langle\ \rangle)$ with $\langle\ \rangle_p$ of type (r, s), $r + s = n - 1$, let $R_p \in \operatorname{Rad}_p$; then $(N, \langle\ \rangle) \pitchfork_p S(r, s)$ if and only if there exists $X_p \in T_pN$ with $\mathrm{II}_p(X, R, R) \neq 0$. Further, the radical is transverse if and only if $\mathrm{II}_p(R, R, R) \neq 0$.*

PROOF. Choose a local frame $e_1, \ldots, e_n \in V_p^\infty(N)$ such that

$$\langle e_i, e_j\rangle_p = \begin{pmatrix} 1 & & & & & 0 & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \vdots \\ & & & & \ddots & & \\ 0 & & & & & -1 & \\ 0 & & & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

so that we may assume $e_n(p) = R_p$. It follows that

$$\det(\langle e_i, e_j\rangle) \equiv \langle e_n, e_n\rangle \mod \mathscr{M}_p^2(N).$$

Thus by Proposition 1, $(N, \langle\ \rangle) \pitchfork_p S(r, s)$ if and only if there exists $X_p \in T_pN$ such that $0 \neq X\langle e_n, e_n\rangle|_p = 2\Box_{X_p} e_n(e_n) = 2\mathrm{II}_p(X, R, R)$. In this case, T_pS is just the kernel of the 1-form $d(\langle e_n, e_n\rangle)_p \in T_pN^*$. So the radical is transverse if and only if $0 \neq R\langle e_n, e_n\rangle|_p = \mathrm{II}_p(R, R, R)$.

Of central interest in this paper is the case where we are given i: $N \to (M, \langle\ \rangle)$ with $(M, \langle\ \rangle)$ nondefinite, *non*singular and $(N, i^*\langle\ \rangle) \pitchfork_p S(r, s)$. The following provides some geometric intuition for II_p in this setting.

PROPOSITION 5. *Given i: $N \to (M, \langle\ \rangle)$ as above, let ∇ denote the classical Levi-Civita connection for $(M, \langle\ \rangle)$ (recall $(M, \langle\ \rangle)$ is assumed to be nondegenerate), and let ${}^N\Box$ denote the dual connection of $(N, i^*\langle\ \rangle)$. If $Y \in V^\infty(M)$ is a vector field everywhere tangent to N and if $X_p, Z_p \in T_pN$ then*

$$ {}^N\Box_{X_p} Y(Z) = \langle \nabla_X Y, Z\rangle_{i(p)}. $$

PROOF. Let X, Y, $Z \in V^\infty(M)$ be vector fields everywhere tangent to M. So there exist $\overline{X}$, $\overline{Y}$, $\overline{Z} \in V^\infty(N)$ which are i-related to X, Y, Z, respectively. Recalling that $[\overline{X}, \overline{Y}] = [X, Y]$ on N, relation $(*)$ implies the result.

Now since $T_pN \cap (T_pN)^\perp = \mathrm{Rad}_p$ we see that II_p is just a fragment of the classical second fundamental form which, due to the peculiarities of nondefinite geometry, is intrinsic to $(N, i^*\langle\ \rangle)$. We now prove the central result of the paper. The reader may wish to consult the addendum to review the notion of fold singularity.

THEOREM 6. *Let i: $N \to (M, \langle\ \rangle)$, $i(p) = q$ be a codimension k immersion with $(N, i^*\langle\ \rangle) \pitchfork_p S(r, s)$ of codimension 1 and transverse radical, then $\bar{i}$: $N^\perp \to M$ has a fold singularity at o_p.*

PROOF. We first deal with the case where N is a hypersurface (i.e. $k = 1$). We may choose a basis of T_qM so that in the associated exponential coordinates $z_1, \ldots, z_m$ for M we have

(1) $i_* T_pN = \mathrm{span}(\partial z_1, \ldots, \partial z_{m-1})$,

(2) mod $\oplus^{m(m+1)/2}[\mathscr{M}_0^2(T_qM)]$,

$$ \langle \partial z_i, \partial z_j\rangle_p \equiv \begin{pmatrix} 1 & & & & & 0 & & \\ & \ddots & & & & & & \\ & & 1 & & & & 0 & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ 0 & & & & & -1 & & \\ & & & & & & 0 & 1 \\ & & 0 & & & & & \\ & & & & & & 1 & 0 \end{pmatrix}. $$

In these coordinates i has the representative

$$ i\colon (\mathbf{R}^{m-1}, 0) \to (\mathbf{R}^m, 0), $$

$$ (x_1, \ldots, x_{m-2}, y) \to (x_1, \ldots, x_{m-2}, y, f(x_1, \ldots, x_{m-2}, y)) $$

with partials f_{x_i} and $f_y \in \mathcal{M}_0(\mathbf{R}^{m-1})$. Thus in the above coordinate frame $i^*\langle\ \rangle$ is represented by h_{ij} where, modulo $\oplus[\mathcal{M}_0^2(\mathbf{R}^{m-1})]$,

$$h_{ij} \equiv \begin{pmatrix} 1 & & & & & 0 & f_{x_1} \\ & \ddots & & & & & \cdot \\ & & 1 & & & & \cdot \\ & & & -1 & & & \cdot \\ & & & & \ddots & & \\ 0 & & & & & -1 & f_{x_{m-1}} \\ f_{x_1} & , & \cdots & , & f_{x_{m-1}} & & 2f_y \end{pmatrix}.$$

Noting that ∂y spans the radical at p we compute $\mathrm{II}_p(\partial y, \partial y, \partial y)$ as

$$\square_{\partial y_p} \partial y(\partial y) = \tfrac{1}{2}\partial y(i^*\langle \partial y, \partial y\rangle)_0 = f_{yy}(0).$$

We see that both hypotheses are satisfied if and only if $f_{yy}(0) \neq 0$ (see Proposition 4).

Now we may describe an orthogonal vector field R in the ∂z_i frame as $(a_1, \dots, a_{m-2}, b - 1, c)$ where a_i, b and $c \in \mathcal{M}_0(\mathbf{R}^{m-1})$. Since $R_q = i_*\partial y$, we compute using Proposition 5 that

$$\square_{\partial y_p} \partial y(\partial y) = -\langle \nabla_{i_*\partial y} i_*\partial y, R\rangle_q = \langle \nabla_{i_*\partial y} R, i_*\partial y\rangle_q = c_y(0).$$

(Recall that $\nabla_{\partial z_{i_q}} \partial z_j = 0$ since we are in exp coordinates.) Thus $c_y(0) = f_{yy}(0) \neq 0$.

Recall that relative to the induced coordinates on TM the map

$$e\colon TM \to M$$
$$v_x \to \exp_x v$$

has $e_*|_{o_x}$ given by the matrix (Id|Id)

$$v_x \to \exp_x v$$

(see Milnor, p. 58). Now observe that we may write $\bar{i}$ as the composite $N^\perp \overset{\bar{j}}{\hookrightarrow} TM \overset{e}{\to} M$, where j is represented by

$$j\colon \mathbf{R}^m, 0 \to \mathbf{R}^{2m}, (0,0), \qquad (x_i, y, \lambda) \to (x_i, y, f, \lambda(a_i, b - 1, c)),$$

relative to the above coordinates. It follows that, modulo $\oplus^m(\mathcal{M}_0^3(\mathbf{R}^m))$, $\bar{i}$ is equivalent to

$$(**) \qquad \left(x_i + \lambda q_i, y - \lambda + \lambda b + \lambda^2\bar{b}, f + \lambda c\right).$$

Now the kernel of $\bar{i}_*|_{o_p}$ is spanned by $\partial y + \partial\lambda|_o$ and the image of $\bar{i}_*|_{o_p}$ is the tangent space to N at p. So if we set $\gamma(t) = (0, \dots, 0, t, t)$ the intrinsic derivative at o_p will be surjective if and only if $(d^2/d^2t)\bar{i}(\gamma(t))|_{t=0}$ does not lie in the image of $\bar{i}_*|_{o_p}$. But this is the same condition as

$$0 \neq \left\langle (d^2/d^2t)\bar{i}(\gamma(t))|_{t=0}, R_p\right\rangle = \left(d^2/d^2t\right)f(\gamma(t)) + tc(\gamma(t))|_{t=0}.$$

However, by the above, this is $3f_{yy}(0) \neq 0$.

We now show that the kernel of $\bar{i}_*|_{o_p}$ is transverse to the singular locus. Viewing $\bar{i}_*$ as a map from $\mathbf{R}^m$ to $\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^m)$ we use (**) to write

$$\bar{i}_* \equiv \begin{pmatrix} * & & * & \\ * & 1 - \lambda b_y & & -1 + b + 2\lambda \bar{b} \\ & f_y + \lambda c_y & & c \end{pmatrix} \quad \mathrm{mod} \bigoplus^{m^2}\left(\mathscr{M}_0^2(\mathbf{R}^m)\right).$$

Since

$$i_*|_{o_p} = \begin{pmatrix} \mathrm{Id} & 0 & \\ 0 & 1 & -1 \\ & 0 & 0 \end{pmatrix},$$

we have that

$$\det(\bar{i}_*) \equiv \left(f + \lambda c_y + c\right) \quad \mathrm{mod}\, \mathscr{M}_0^2(\mathbf{R}^m).$$

Because the kernel of $(d(\det(\bar{i}_*)|_{o_p})) \in T^*_{o_p}(N^\perp)$ identifies with the tangent space to the singular locus of $\bar{i}$, we see that the pairing of $\ker \bar{i}_*$ and $d(\det(\bar{i}_*))$ at o_p is $(\partial y + \partial\lambda)(f_y + c_y + c)_0 \neq 0$.

We deal with the higher codimension case $k > 1$ by enlarging matters to the above hypersurface setting. Choose a smooth local foliation of N so that the leaf at p corresponds to the *singular locus* S. At each point of N sufficiently near p the leaves inherit a nondegenerate metric (recall Rad is transverse). Thus near p in N we may modify the Gram-Schmidt algorithm to decompose the tangent space to M as $T\,\mathrm{Leaf} \oplus R \oplus Q \oplus P$ where

(1) $(T\,\mathrm{Leaf})^\perp = R \oplus Q \oplus P$,

(2) P is of dimension $k - 1$, nondegenerate, and orthogonal to $R \oplus Q$,

(3) $R = \mathrm{Rad}$ over S,

(4) $R \oplus Q$ is a type $(1,1)$ 2-plane bundle in TM.

Thus P is a rank $k - 1$ vector bundle over a neighborhood U of p in N, which is contained in $N^\perp$ over U. Now the map $i_{\mathrm{ext}}\colon P \to M$ defined by $w_x \to \exp_{i(x)} w$ is an immersion at o_p. Since $(i_{\mathrm{ext}})_* T_{o_p}P = T_pS \oplus \mathrm{Rad}_p \oplus P_p$ we see that $(P, i^*_{\mathrm{ext}}\langle\ \rangle)$ is metrically singular at o_p with $r + s = m - 1$ and its radical at o_p agrees with Rad_p. It is easy to see that

$$P\mathrm{II}_{o_p}(R, R, R) = \mathrm{II}_p(R, R, R) \neq 0.$$

Further, by construction the two geodesic tube maps, $\bar{i}$ and $(\overline{i_{\mathrm{ext}}})$ agree over the neighborhood U of p. Thus, by the above $k = 1$ result, we are finished.

EXAMPLE. We will illustrate the above proposition in a simple and readily visualized setting; namely, where $n = 1$, $m = 2$ and the ambient metric is flat. Consider

$$i\colon \mathbf{R} \to \left(\mathbf{R}^2, \langle\ \rangle \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad t \to (t, t^r/r),$$

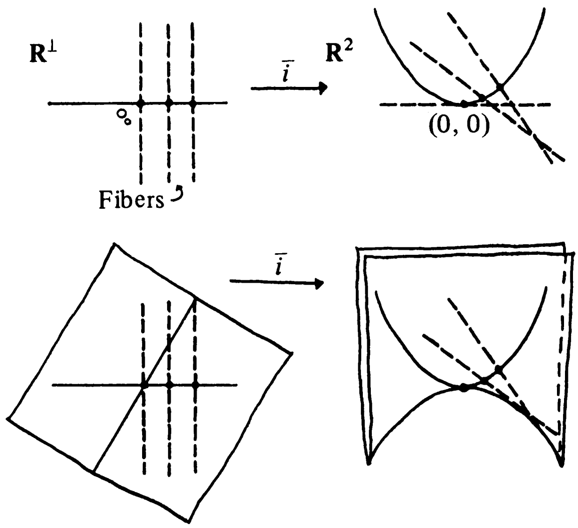
where $r = 2$, or 3. We have $i^*\langle\ \rangle = 2t^{r-1} dt \circledcirc dt$, so that if we set $\partial t = R$ at 0, then we have

$$\mathrm{II}(R, R, R)_0 = \tfrac{1}{2}\partial t\langle \partial t, \partial t\rangle_0.$$

Hence the metric singularity is transverse and radical transverse only when $r = 2$. Further, $\bar{i}\colon \mathbf{R}^{\perp} \simeq \mathbf{R} \times \mathbf{R} \to \mathbf{R}^2$ is given by $t, \lambda \to (t, t^r/r) + \lambda(1, -t^{r-1})$ and

$$(\bar{i})_* \simeq \begin{pmatrix} 1 & 1 \\ t^{r-1} - \lambda(r-1)t^{r-2} & -t^{r-1} \end{pmatrix}.$$

Hence $\det(\bar{i})_*$ is given by $\lambda - 2t$ if $r = 2$ and by $2\lambda t - 2t^2$ if $r = 3$. $\ker(\bar{i})_*|_{0,0}$ is given by $\partial t - \partial\lambda$ in both cases. Thus, $j^1(\bar{i})$ is transverse to ${}_1S$ only when $r = 2$. We may visualize the $r = 2$ case, as below.



Addendum: Fold singularities. Let $g\colon (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ be the germ of a smooth map and let ${}_1S \to J^1(\mathbf{R}^m, \mathbf{R}^m)$ denote the codimension one submanifold of jets with rank one less than maximal. g has a *fold* at 0 if j^1g is transverse to ${}_1S$ at 0. And the *fold locus* $(j^1g)^{-1}({}_1S)$ is transverse to the kernel of $(g_*)_0$ in $T_0\mathbf{R}^m$. Recall that the first transversality condition may be stated as the surjectivity of the intrinsic derivative $D(g_*)_0\colon T_0\mathbf{R}^m \to \operatorname{Hom}(\ker(g_*)_0, \operatorname{coker}(g_*)_0)$. In the case at hand, both the ker and coker are one-dimensional. Thus this condition will be satisfied if we can find a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to (\mathbf{R}^m, 0)$ such that $\dot{\gamma}(0)$ spans $\ker(g_*)$ and $(d^2/d^2t)g(\gamma(t))|_{t=0}$ does not lie in the image of $(g_*)_0$ (i.e. $Dg_*(\ker g_*) \not\equiv 0$).

Up to the action of $\operatorname{Diff}_0^\infty(\mathbf{R}^m) \times \operatorname{Diff}_0^\infty(\mathbf{R}^m)$ a fold singularity has the germ

$$\mathbf{R}^m, 0 \to \mathbf{R}^m, 0, (\omega_1, \ldots, \omega_m) \to (\omega_1, \ldots, \omega_{m-1}, \omega_m^2)$$

as a representative.

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