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SHORTER NOTES

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A FORMALLY NORMAL OPERATOR HAVING NO NORMAL EXTENSION

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ABSTRACT. We give an example of a formally normal operator N satisfying $\dim \mathscr{D}(N^*)/\mathscr{D}(N) = 1$ which has no normal extension in any larger Hilbert space.

For a linear operator T on a Hilbert space $\mathscr{H}$, we denote by $\mathscr{D}(T)$ its domain. A *formally normal operator* N on $\mathscr{H}$ is a densely defined closed linear operator on $\mathscr{H}$ such that $\mathscr{D}(N) \subseteq \mathscr{D}(N^*)$ and $\|Nx\| = \|N^*x\|$ for all $x \in \mathscr{D}(N)$. A *normal operator* is a formally normal operator N on $\mathscr{H}$ satisfying $\mathscr{D}(N) = \mathscr{D}(N^*)$.

The first example of a formally normal operator N which has no normal extension in a larger Hilbert space is due to E. A. Coddington [**1**]. Coddington's example is minimal in the sense that $\dim \mathscr{D}(N^*)/\mathscr{D}(N) = 1$. The aim of this note is to present a very simple example of that kind. For let S be the unilateral shift on the Hardy space $\mathscr{H} = H^2(\mathbb{T})$, and let $A := i(S+I)(S-I)^{-1}$. Set $B := S + S^*$.

PROPOSITION. *The operator $N := A + iB$ is a formally normal operator on $\mathscr{H}$ which has no normal extension in a possibly larger Hilbert space. Moreover,* $\dim \mathscr{D}(N^*)/\mathscr{D}(N) = 1$.

PROOF. First we show that $BA \subseteq A^*B$. Let $x \in \mathscr{D}(A)$. Then $x = (S-I)y$ for some $y \in \mathscr{H}$ and $B(A-i)x = 2iBy$. Let $P := z^0 \otimes z^0$. Since $(I-P)BS = SB$ and $z^0 \in \ker(A^* - i)$, $Bx = (S-I)By + PBSy \in \mathscr{D}(A^*)$ and $(A^* - i)Bx = (A-i)(S-I)By = 2iBy = B(A-i)x$. This proves $BA \subseteq A^*B$.

From $BA \subseteq A^*B$ and $B = B^*$, it follows that $\langle Ax, Bx\rangle = \langle Bx, Ax\rangle$ for $x \in \mathscr{D}(A)$. Since $N^* = A^* - iB$ and $\mathscr{D}(N) \subseteq \mathscr{D}(N^*)$, the latter implies $\|Nx\|^2 = \|Ax\|^2 + \|Bx\|^2 = \|N^*x\|^2$ for $x \in \mathscr{D}(N) = \mathscr{D}(A)$. That is, N is formally normal. Since A has deficiency indices $(0,1)$, $\dim \mathscr{D}(N^*)/\mathscr{D}(N) = 1$.

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Assume that there exists a normal extension N_1 of N in a Hilbert space $\mathscr{H}_1 \supseteq \mathscr{H}$. Let U_1 denote the Cayley transform of the selfadjoint operator $A_1 := \overline{\operatorname{Re} N_1}$ on $\mathscr{H}_1$. From $A_1 \supseteq A$ and $(A - i)\mathscr{D}(A) = \mathscr{H}$ it follows that $U_1 \upharpoonright \mathscr{H} = S$. Clearly, $B \subseteq B_1 := \overline{\operatorname{Im} N_1}$. Because N is normal, $U_1 B_1 \subseteq B_1 U_1$. Since $B = B_1 \upharpoonright \mathscr{H}$ and $S = U_1 \upharpoonright \mathscr{H}$, this implies $SB = BS$. Since $B = S + S^*$ by definition, this is a contradiction.

REMARKS. 1. A slight modification of the above proof shows the following: Suppose A is a maximal symmetric operator on a Hilbert space $\mathscr{H}$. Suppose B is a bounded selfadjoint operator on H such that $BA \subseteq A^*B$, but $BA \not\subseteq AB$. Then $N := A + iB$ is a formally normal operator which cannot be extended to a normal operator in any larger Hilbert space.

2. In the above example, B can be replaced by an arbitrary bounded selfadjoint Toeplitz operator B_φ with symbol $\varphi \neq 0$ in $L^\infty(\mathbb{T})$. (By Example 1.10 in [**2**], $B_\varphi A \subseteq A^* B_\varphi$ and $B_\varphi A \not\subseteq AB_\varphi$.)

3. Suppose A and B are selfadjoint operators which commute on a common core $\mathscr{D}$ and for which the spectral projections do not commute. Then $N := \overline{(A + iB) \upharpoonright \mathscr{D}}$ is formally normal and has no normal extension in a possibly larger Hilbert space [**3**, Lemma 1.5]. But it seems to be more difficult in general to calculate $\dim \mathscr{D}(N^*)/\mathscr{D}(N)$ in that case.

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