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GENERALIZATION OF TWO RESULTS OF THE THEORY OF UNIFORM DISTRIBUTION

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ABSTRACT. For a sequence $x_1,\dots,x_N$ of points in $[0,1]$ and a sequence $p_1,\dots,p_N$ ($p_1+p_2+\cdots+p_N=1$) of nonnegative numbers, define the distribution function

$$g(x) = x - \sum_{x_k<x} p_k.$$

Let φ be an increasing function on $[0,1]$ and $\varphi(0)=0$. The main result of the paper is

$$F(D_N) \leqslant \int_0^1 \varphi(|g(x)|)\,dx \leqslant \varphi(D_N),$$

where D_N is the supremum norm of g on $[0,1]$ and F is the antiderivative of φ with $F(0)=0$. This result generalizes and improves an estimate of Niederreiter [**1**] for the L^2 discrepancy of the sequence $x_1,\dots,x_N$. Applying the above inequality we also obtain a new criterion for uniform distribution modulo one.

1. Introduction. Let us recall some definitions and results of the theory of uniform distribution modulo one.

DEFINITION A. Let $x_1, x_2,\dots,x_N$ be a finite sequence in the interval $[0,1]$. The number

$$D_N^{(p)} = \left(\int_0^1 \left| x - \frac{1}{N}\sum_{\substack{1\leqslant k\leqslant N\\ x_k<x}} 1\right|^p dx\right)^{1/p}, \qquad 0<p\leqslant\infty,$$

is called the L^p discrepancy of the given sequence.

In what follows, we shall write D_N instead of $D_N^{(\infty)}$.

H. Niederreiter proved the following theorem using the well-known inequality of LeVeque.

THEOREM A (NIEDERREITER [**1**]). *For any sequence $x_1, x_2,\dots,x_N$ in $[0,1]$ we have*

$$\frac{1}{\sqrt{12}} D_N^{3/2} \leqslant D_N^{(2)} \leqslant D_N. \tag{1}$$

DEFINITION B. Let $\sigma=(x_n)$ be an infinite sequence in $[0,1]$. For the infinite sequence σ, the L^p discrepancy $D_N^{(p)}(\sigma)$ is defined to be the L^p discrepancy of the initial segment formed by the first N terms of σ.

Again, we shall write $D_N(\sigma)$ instead of $D_N^{(\infty)}(\sigma)$.

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DEFINITION C. An infinite sequence σ in $[0,1]$ is said to be uniformly distributed (in the sense of Weyl) if

$$\lim_{N\to\infty} D_N(\sigma) = 0.$$

The following criterion for uniform distribution is well known.

THEOREM B (SOBOL [**2**, P. 115]). *Suppose* $0 < p < \infty$. *Then an infinite sequence* σ *in* $[0,1]$ *is uniformly distributed if and only if*

$$\lim_{N\to\infty} D_N^{(p)}(\sigma) = 0.$$

In §2 we generalize and improve Theorem A. In §3 we generalize Theorem B.

2. Generalization of Theorem A. Suppose we are given a finite sequence $x_1, x_2,\dots,x_N$ in $[0,1]$ and a finite sequence $p_1, p_2,\dots,p_N$ of nonnegative numbers. We call the numbers $p_1, p_2,\dots,p_N$ weights of the numbers $x_1, x_2,\dots,x_N$, respectively. Let us define the functions h and g on $[0,1]$ by

$$h(x) = \sum_{\substack{1\leqslant k\leqslant N\\ x_k<x}} p_k \tag{2}$$

and

$$g(x) = x - h(x). \tag{3}$$

Obviously, the function h is increasing on $[0,1]$ and the function g is piecewise linear on $[0,1]$.

DEFINITION 1. The number

$$D_N^{(p)} = \left(\int_0^1 |g(x)|^p\,dx\right)^{1/p}, \qquad 0 < p \leqslant \infty,$$

is said to be the L^p discrepancy of the sequence $x_1, x_2,\dots,x_N$ with respect to the weights $p_1, p_2,\dots,p_N$.

Instead of $D_N^{(\infty)}$ we shall write D_N. Evidently,

$$D_N = \sup_{0\leqslant x\leqslant 1} |g(x)|. \tag{4}$$

Comparing Definition A with Definition 1 we see that the L^p discrepancy of the sequence $x_1, x_2,\dots,x_N$ is equal to the L^p discrepancy of this sequence with respect to the weights $p_1 = p_2 = \cdots = p_N = 1/N$.

THEOREM 1. *Let* φ *be an increasing function on* $[0,1]$, $\varphi(0) = 0$ *and*

$$F(x) = \int_0^x \varphi(t)\,dt. \tag{5}$$

Then for any sequence $x_1, x_2,\dots,x_N$ *in* $[0,1]$ *and any weights* $p_1, p_2,\dots,p_N$ *with*

$$\sum_{k=1}^N p_k = 1 \tag{6}$$

we have

$$F(D_N) \leqslant \int_0^1 \varphi(|g(x)|)\,dx \leqslant \varphi(D_N), \tag{7}$$

where the function g *is defined by* (3).

PROOF. The second inequality in (7) is obvious. It holds true because φ increases on $[0,1]$. Now, we shall prove the first inequality in (7). Let a be an arbitrary real number with

$$0 < a < D_N. \tag{8}$$

First we shall prove the following inequality.

$$F(a) \leqslant \int_0^1 \varphi(|g(x)|)\,dx. \tag{9}$$

It follows from (4) and (8) that there exists a number $x_0 \in [0,1]$ such that

$$|g(x_0)| > a.$$

According to the definition of g, the above inequality can be written as

$$|x_0 - h(x_0)| > a.$$

Hence, there are two possible cases:

$$h(x_0) > x_0 + a \quad \text{or} \quad h(x_0) < x_0 - a.$$

Further, we shall prove (9) in the first case only because it can similarly be proved in the second case as well.

Suppose $h(x_0) > x_0 + a$. Then from (2) and (6), we have

$$[x_0, x_0 + a] \subset [0,1]. \tag{10}$$

Since φ is an increasing function on $[0,1]$ and $\varphi(0) = 0$, it follows that the inequality $\varphi(|g(x)|) \geqslant 0$ holds for every $x \in [0,1]$. Hence, we obtain from (10)

$$\int_0^1 \varphi(|g(x)|)\,dx \geqslant \int_{x_0}^{x_0+a} \varphi(|g(x)|)\,dx. \tag{11}$$

Now suppose that $x \in [x_0, x_0 + a]$. Since the function h increases on $[0,1]$, we deduce

$$g(x) \leqslant x - h(x_0) < x - x_0 - a \leqslant 0.$$

Therefore,

$$|g(x)| = -g(x) > x_0 + a - x.$$

Hence, using (11) and (5) we get

$$\int_0^1 \varphi(|g(x)|)\,dx \geqslant \int_{x_0}^{x_0+a} \varphi(x_0 + a - x)\,dx = F(a).$$

Thus, (9) is proved in the first case.

Now it follows from (9) that

$$\sup_{0<a<D_N} F(a) \leqslant \int_0^1 \varphi(|g(x)|)\,dx. \tag{12}$$

But the function F is increasing on $[0,1]$ because $\varphi(x) \geqslant 0$ for every $x \in [0,1]$. Hence, F is increasing on $[0, D_N]$, too, because $0 < D_N \leqslant 1$. Therefore,

$$\sup_{0<a<D_N} F(a) = F(D_N). \tag{13}$$

Finally, the first inequality in (7) follows from (12) and (13). Theorem 1 is proved.

Setting $\varphi(x) = x^p$ $(0 < p < \infty)$ in Theorem 1, we immediately obtain

COROLLARY 1. *Suppose* $0 < p < \infty$. *Then for any sequence* $x_1, x_2, \ldots, x_N$ *in* $[0,1]$ *and any weights* $p_1, p_2, \ldots, p_N$ *with* (6) *we have*

$$\frac{1}{(p+1)^{1/p}} D_N^{1+1/p} \leqslant D_N^{(p)} \leqslant D_N. \tag{14}$$

REMARK 1. From (14) we get the following estimate for $p = 2$,

$$\frac{1}{\sqrt{3}} D_N^{3/2} \leqslant D_N^{(2)} \leqslant D_N. \tag{15}$$

It is easy to see that (15) improves estimate (1) of Niederreiter.

REMARK 2. The first inequality in (7) changes into an equality if $x_1 = x_2 = \cdots = x_N = 0$.

3. Generalization of Theorem B. Suppose we are given two infinite triangular matrices $X = (x_k^{(n)})$ and $P = (p_k^{(n)})$ with $0 \leqslant x_k^{(n)} \leqslant 1$ and $p_k^{(n)} \geqslant 0$ $(n = 1, 2, \ldots;$ $k = 1, \ldots, n)$. We call the matrix P a weight matrix of the matrix X.

DEFINITION 2. Suppose $0 < p \leqslant \infty$. The L^p discrepancy $D_n^{(p)}(X, P)$ is defined to be the L^p discrepancy of the sequence $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ with respect to the weights $p_1^{(n)}, p_2^{(n)}, \ldots, p_n^{(n)}$, i.e.

$$D_n^{(p)}(X, P) = \left(\int_0^1 |g_n(x)|^p \, dx\right)^{1/p}, \tag{16}$$

where

$$g_n(x) = x - \sum_{\substack{1 \leqslant k \leqslant n \\ x_k^{(n)} < x}} p_k^{(n)}. \tag{17}$$

DEFINITION 3 (SEE [3]). The matrix X is said to be uniformly distributed with respect to the weight matrix P if

$$\lim_{n \to \infty} D_n(X, P) = 0. \tag{18}$$

DEFINITION 4. Let φ be a function defined on $[0,1]$. We call φ a basic function if it satisfies the following three conditions:

(i) φ is increasing on $[0,1]$,

(ii) $\lim_{x \to 0^+} \varphi(x) = 0$,

(iii) $\varphi(x) = 0$ if and only if $x = 0$.

LEMMA 1. *Let* φ *be a basic function. Then the function* F *defined by* (5) *is a basic function as well.*

PROOF. From (5), (i) and (iii), we deduce

$$F(x_2) - F(x_1) \geqslant \frac{x_2 - x_1}{2} \varphi\left(\frac{x_1 + x_2}{2}\right) > 0$$

for all x_1 and x_2 with $0 \leqslant x_1 < x_2 \leqslant 1$. Therefore, F is strictly increasing on $[0,1]$. Hence, for every $x \in (0,1]$ we have

$$0 = F(0) < F(x) \leqslant x\varphi(x).$$

Passing to the limit as $x \to 0^+$ in this inequality, we get $\lim_{x \to 0^+} F(x) = 0$.

LEMMA 2. *Let φ be a basic function. Then the matrix is uniformly distributed with respect to the weight matrix if and only if*

$$\lim_{n \to \infty} \varphi(D_n(X, P)) = 0. \tag{19}$$

PROOF. The necessity follows immediately from (ii) and Definition 3. Now suppose that (19) holds, but (18) does not hold. Then there exists a positive number ε_0 such that the inequality

$$D_n(X, P) \geqslant \varepsilon_0 \tag{20}$$

holds for infinitely many values of n. It follows from (20) and (i) that

$$\varphi(D_n(X, P)) \geqslant \varphi(\varepsilon_0). \tag{21}$$

From (19), (21) and (iii), we deduce

$$0 = \lim_{n \to \infty} \varphi(D_n(X, P)) \geqslant \varphi(\varepsilon_0) > 0,$$

which is a contradiction. Therefore, if (19) holds then (18) holds, too, i.e. X is uniformly distributed with respect to P.

The following criterion for uniform distribution is a generalization of Theorem B.

THEOREM 2. *Let φ be a basic function and let P be a weight matrix with*

$$\sum_{k=1}^{n} p_k^{(n)} = 1 \qquad (n = 1, 2, \ldots). \tag{22}$$

Then a matrix X is uniformly distributed with respect to the weight matrix P if and only if

$$\lim_{n \to \infty} \int_0^1 \varphi(|g_n(x)|)\, dx = 0,$$

where $g_n(x)$ is defined by (17).

PROOF. By Theorem 1 we have

$$F(D_n(X, P)) \leqslant \int_0^1 \varphi(|g_n(x)|)\, dx \leqslant \varphi(D_n(X, P)). \tag{23}$$

Since φ is a basic function, it follows from Lemma 1 that F is a basic function, too. Now, the assertion follows from (23) and Lemma 2.

Setting $\varphi(x) = x^p$ $(0 < p < \infty)$ in Theorem 2, we immediately obtain

COROLLARY 2. *Let $0 < p < \infty$ and P be a weight matrix with* (22). *Then a matrix X is uniformly distributed with respect to the weight matrix P if and only if*

$$\lim_{n \to \infty} D_n^{(p)}(X, P) = 0.$$

REMARK 3. It is easy to see that Corollary 2 is a generalization of Theorem B. Indeed, let $\sigma = (x_n)$ be an infinite sequence in $[0, 1]$. Applying Corollary 2 for the matrices $X = (x_k^{(n)})$ and $P = (p_k^{(n)})$ with $x_k^{(n)} = x_k$ and $p_k^{(n)} = 1/n$ $(n = 1, 2, \ldots;$ $k = 1, \ldots, n)$ we get Theorem B.

4. Final remark. Theorem 2 shows that as a measure of the distribution of a matrix X with respect to a weight matrix P, alongside with the L^p discrepancy $D_n^{(p)}(X, P)$, one may use the φ-discrepancy

$$D_n^{(\varphi)}(X, P) = \int_0^1 \varphi\left(|g_n(x)|\right) dx,$$

where φ is a basic function and $g_n(x)$ is defined by (17).

Similarly, as a measure of the distribution of a sequence $x_1, x_2, \dots, x_N$ in $[0, 1]$ with respect to the weights $p_1, p_2, \dots, p_N$, alongside with the L^p discrepancy $D_N^{(p)}$, one can use the φ-discrepancy

$$D_N^{(\varphi)} = \int_0^1 \varphi\left(|g(x)|\right) dx,$$

where φ is a basic function, too, and $g(x)$ is defined by (3).

References

1. H. Niederreiter, *Application of diophantine approximation to numerical integration*, Diophantine Approximation and its Applications (C. F. Osgood, ed.), Academic Press, New York, 1973, pp. 129–199.

2. I. M. Sobol, *Multidimensional quadrature formulae and Haar functions*, Nauka, Moscow, 1969.

3. P. D. Proinov, *Note on the convergence of the general quadrature process with positive weights*, Constructive Function Theory'77 (Bl. Sendov and D. Vačov, eds.), Sofia, 1980, pp. 121–125.

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