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CORRIGENDUM TO "EMBEDDINGS IN $G(1,3)$"

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As was pointed out by N. Goldstein in MR **84i**, Lemma 2.2 in [**3**] is incorrect. We give here a revised version of the main theorem in [**3**]. In the proof we avoid Lemma 2.2 and use instead [**2**, Theorem 2.3 and Corollary 2.4], [**1**, Corollary IV.18], and Castelnuovo's bound on the genus of a curve in $\mathbf{P}^n$. In the revised theorem the restriction to surfaces that are not projections from a higher $\mathbf{P}^n$ is removed.

THEOREM. *Let Y be a nonsingular surface in $G \subset \mathbf{P}^5$ of degree $\leqslant 8$, and let (a, b) be its class in the Chow ring $A(G)$ of G. Then one of the following holds*:

(i) $d = 1$, $(a, b) = (1, 0)$, $Y = \mathbf{P}^2$;

(ii) $d = 2$, $(a, b) = (1, 1)$, $Y = F_0$;

(iii) $d = 3$, $(a, b) = (2, 1)$, $Y = F_1$;

(iv) $d = 4$, *and either* $(a, b) = (2, 2)$ *and* $Y = F_0$, F_2, *or the del Pezzo* S_4, *or else* $(a, b) = (1, 3)$ *and* $Y =$ *the Veronese surface*;

(v) $d = 5$, $(a, b) = (2, 3)$ *and* $Y = F_e$ *with* 3 *or* 7 *points blown up*;

(vi) $d = 6$, $(a, b) = (3, 3)$, *and either* $Y = F_e$ *with* 2 *or* 6 *points blown up*, *or* Y *is a geometrically ruled surface with* $p_a = -1$, *or* $Y = G \cap \mathbf{P}^4 \cap S_3$ *and is a* $K3$ *surface*;

(vii) $d = 7$, *and either* Y *is geometrically ruled with* $p_a = -3$, *or* Y *is ruled with* 2 *points blown up with* $p_a = -3$, *or* Y *is ruled with* 4 *or* 6 *points blown up with* $p_a = -1$, *or* $Y = F_e$ *with* 8 *points blown up*, *or* $Y =$ *the cubic surface with* 5 *points blown up*, *or* $K^2 = -12 + 6p_a$;

(viii) $d = 8$, *and either* $(a, b) = (4, 4)$, *and* $Y = F_e$ *with* 6 *or* 10 *points blown up*, *or* Y *is geometrically ruled with* $p_a = -3$, *or* Y *is ruled with* 4 *points blown up with* $p_a = -1$, *or* Y *is a complete intersection of three quadrics*, *or* $Y = G \cap \mathbf{P}^4 \cap S_4$ *is a surface of general type*; *or* $(a, b) = (2, 6)$ *and* Y *is geometrically ruled with* $p_a = -1$; *or* $(a, b) = (3, 5)$ *and* Y *is ruled with* 3 *points blown up with* $p_a = -1$.

To facilitate the reading of the proof, we list here some facts we use.

I. CASTELNUOVO'S BOUND ON THE GENUS. Let C be a nonsingular curve of degree d, $C \subset \mathbf{P}^n$, not lying in any $\mathbf{P}^{n-1}$. Then $g \leqslant m(m-1)(n-1)/2 + m\varepsilon$ where $m = [d - 1/n - 1]$ *and* $\varepsilon = (d-1) - m(n-1)$.

II. CRITERION FOR A SURFACE TO BE RULED [**1**, Corollary VI.18]. S is ruled if and only if there is a curve $C \subset S$, not an exceptional divisor, such that $C \cdot K < 0$.

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III. CURVES ON NONRATIONAL RULED SURFACES. (1) [**2**, Theorem 2.3]: Let π: $Y \to T$ be a nonrational ruled surface, $C \subset Y$ an irreducible curve, and m the degree of π: $C \to T$, with $m > 1$. Then

$$C^2 \leqslant \frac{2m}{m-1}(g(C) - 1).$$

(2) [**2**, Corollary 2.4]: Let C and Y be as above. Then either

(a) $C \cong T$ and the embedding of $C \subset Y$ is equivalent to a section of the geometrically ruled surface π: $\mathbf{P}(E) \to T$ and $(C^2)_Y = (T_0^2)_{\mathbf{P}(E)}$, or

(b) $C^2 \leqslant 4g(C) - 4$.

IV. FURTHER EQUATIONS. With the notation as in [**3**, p. 584], let s be the number of points in X with $r_p = 1$, t the number of points in X with $r_p = 2$ and $\varphi^{-1}(p) = E$, one exceptional divisor, and u the number of points in X with $r_p = 2$ and $\varphi^{-1}(p) = E_1 \cup E_2$, two exceptional divisors. Assume that for all $p \in X$, $r_p \leqslant 2$. Then

(3) $X^2 = m(2n - me) = C^2 + s + 4t + 5u$,

(4) $2g(X) - 2 = X^2 - X^2/m - 2m(p_a(Y_0) + 1)$,

(5) $g(X) = g(C) + t + u$,

(6) $K_{Y_0}^2 = K_Y^2 + r$,

(7) $r = s + t + 2u$.

PROOF OF THEOREM. (i) and (ii) are obvious.

(iii) If $d = 3$, then $Y \subset \mathbf{P}^4$; by I $g(C) = 0$, $HK = -5$; II implies that Y is ruled, and III implies $p_a = 0$, and the proof in [**3**] for $d = 3$ applies.

(iv) If $d = 4$ and $Y \not\subset \mathbf{P}^4$, then, by I, $g(C) = 0$, $HK = -6$; II implies that Y is ruled, and III implies that $p_a = 0$, and the proof in [**3**] for $d = 4$ applies.

(v) If $d = 5$ and $Y \not\subset \mathbf{P}^4$, then, by I, $g(C) \leqslant 1$. Suppose $g(C) = 0$. Then $KH = -7$ and Y is a rational surface by II and III, but [**3**, (2)] leads to a contradiction. Therefore $g(C) = 1$, $KH = -5$, and II implies that Y is ruled. If $p_a \neq 0$, then by III(2), $p_a = -1$, C is a section, $X^2 = C^2$, and Y is geometrically ruled. Hence, $K^2 = 0$, and this contradicts [**3**, (2)]. Therefore $p_a = 0$, and the proof in [**3**] for $d = 5$, $Y \not\subset \mathbf{P}^4$ applies.

If $d = 5$ and $Y \subset \mathbf{P}^4$, $g(C) = 0$ or 2 by [**3**, Proposition 1.2]. Suppose $g(C) = 0$. Then $KH = -7$ and Y is a rational ruled surface by II and III. Finally, by [**3**, (3) and (2)] we get a contradiction. Hence, $g(C) = 2$, $KH = -3$, and Y is ruled. If $p_a \neq 0$, then, by III, $p_a = -2$, $K^2 = -8$, and this contradicts [**3**, (2)]. Hence, $g(C) = 2$, $p_a = 0$, and the proof in [**3**] for $d = 5$, $Y \subset \mathbf{P}^4$ applies.

(vi) If $d = 6$ and $Y \not\subset \mathbf{P}^4$, then, by I, $g(C) \leqslant 2$. If $g(C) = 0$, then $KH = -8$, and II and III imply that Y is a rational ruled surface, and this contradicts [**3**, (2)]. If $g(C) = 1$, then $KH = -6$, and Y is ruled. If $p_a = 0$, then by [**3**, Proposition 1.3] $m \leqslant 3$ and $r_p = 1$, and by [**3**, (2)] $K^2 = 7$ or 6. If $K^2 = 7$, then, by (3) and (6), $X^2 = 7$; by (5), $g(X) = g(C) = 1$; and by (4), $0 = 7 - 7/m - 2m$, which is impossible. Hence, $K^2 = 6$, $(a, b) = (3,3)$, and, by (4), $0 = 8 - 8/m - 2m$, which implies $m = 2$. If $p_a \neq 0$, then, by II and III, Y is geometrically ruled with $p_a = -1$, $m = 1$, $e = 0$, $n = 3$, and $(a, b) = (3,3)$. If $g(C) = 2$ with $p_a \neq 0$, then, by II and III, Y is geometrically ruled with $p_a = -2$, and this contradicts [**3**, (2)]. Hence, if $g(C) = 2$, then Y is a rational ruled surface and the proof in [**3**] for $d = 6$ applies.

(vii) If $d = 7$ and $Y \not\subset \mathbf{P}^4$, then, by I, $g(C) \leqslant 3$. If $g(C) = 0$, then $KH = -9$ and Y is a rational ruled surface, which contradicts [**3**, (2)]. If $g(C) = 1$, then $KH = -7$, and if $p_a = 0$, then by [**3**, (2)], $(a, b) = (3,4)$ and Y is geometrically ruled, and, by (4), $2g(C) - 2 = 0 = 7 - 7/m - 2m$, which is impossible. Hence, if $g(C) = 1$, then $p_a \neq 0$, and, by III, $p_a = -1$, $K^2 = 0$, and this contradicts [**3**, (2)].

If $g(C) = 2$ and $p_a = 0$, then by [**3**, (2)], $K^2 = 6$ or 4, and, by [**3**, Proposition 1.3], $m \leqslant 3$, whence $r = 1$. Hence, by (3) $X^2 = 9$ or 11 and, by (4), $2 = 9 - 9/m - 2m$ or $2 = 11 - 11/m - 2m$, and both are impossible. Hence, if$g(C) = 2$, then $p_a \neq 0$, and, by II and III, Y is geometrically ruled with $p_a = -2$, $C^2 = 7 = 2n - e$, and $T_0 \cdot C = n - e \geqslant 5$, since T_0 is a nonsingular curve of genus 2; but this contradicts the fact that $e \geqslant p_a = -2$.

Hence, if $d = 7$, then $g(C) = 3$ and $KH = -3$. If $p_a \neq 0$, then, by III, $m \leqslant 2$. If $m = 1$, then $r_p = 1$, $g(X) = g(C)$, and, by (4), $p_a = -3$. Hence, by [**3**, (2)], Y is of type $(2,5)$ and is geometrically ruled with $e = -1$, $n = 3$, or of type $(3,4)$ with $K^2 = -18$. If $m = 2$, by (3)–(5), we get $9 + 8p_a = s + u \geqslant 0$. Therefore $p_a = -1$ and, by [**3**, (2)], $K^2 = -4$ or -6. If $p_a = 0$, then the proof in [**3**] for $d = 7$ applies.

(viii) If $d = 8$ and $Y \not\subset \mathbf{P}^4$, then, by I, $g(C) \leqslant 5$. Suppose $g(C) = 0$. Then $p_a = 0$ and [**3**, (2)] leads us to a contradiction. If $g(C) = 1$, then by [**3**, (2)], $p_a < 0$, and, by III, $p_a = -1$; hence $K^2 \leqslant 0$, and this contradicts [**3**, (2)]. If $g(C) = 2$ with $p_a = 0$, then, by [**3**, (2)], $K^2 = 6$ or 7, and by [**3**, Proposition 1.3], $m < 4$; hence, $r_p = 1$, and, by (4), $2 = 10 - 10/m - 2m$ or $2 = 9 - 9/m - 2m$, and both are impossible. Hence, if $g(C) = 2$, then $p_a < 0$, and, by III, C is a section, Y is geometrically ruled with $p_a = -2$, and $K^2 = -8$; but this is impossible by [**3**, (2)]. Therefore if $d = 8$ and $Y \not\subset \mathbf{P}^4$, then $3 \leqslant g(C) \leqslant 5$.

If $g(C) = 3$ with $p_a = 0$, then, by [**3**, Proposition 1.3], $m \leqslant 4$. If $m < 4$, then $r_p = 1$, and by (5), $g(X) = 3$, and, by (4), $4 = 8 + r - (8 + r)/m - 2m$; but, by [**3**, (2)], $K^2 = 6, 3$, or 2, which are impossible. If $m = 4$, then $r_p \leqslant 2$, and, by (3)–(5), $3s + 4t + 7u = 24$, and, by (7), $r = 6$ or 7 are the only solutions. Hence, by [**3**, (2)], $r = 6$, $K^2 = 2$, and $(a, b) = (4,4)$. If $g(C) = 3$ with $p_a < 0$ and Y is geometrically ruled, then, by [**3**, (2)], $p_a = -1$ and $(a, b) = (2,6)$ or $p_a = -3$ and $(a, b) = (4,4)$. Since Y is assumed geometrically ruled, we get from (3) and (4) that $p_a = -1$ implies $4 = 8 - 8/m$; hence, $m = 2$, $e = -1$, $n = 1$, and $p_a = -3$ imply $4 = 8 - 8/m + 4m$; hence, $m = 1$, $e = 0$, $n = 4$. If $g(C) = 3$ with $p_a < 0$ and Y not geometrically ruled, then, by III, $m = 2$ and, by (3)–(5), $8 + 8p_a = s + u \geqslant 0$. Hence, $p_a = -1$ and Y is either of type $(3,5)$ with $K^2 = -3$, or of type $(4,4)$ with $K^2 = -4$. If $g(C) > 3$, then the proof in [**3**] for $d = 8$ applies.

References

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