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NUMERICAL RADIUS-ATTAINING OPERATORS ON $C(K)$

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ABSTRACT. Using a construction due to Johnson and Wolfe, we show that the numerical radius-attaining operators from $C(K)$ into itself are dense in the space of all operators, where K is a compact Hausdorff space.

Let X be a Banach space, $L(X)$ the Banach space of bounded linear operators from X into X, and $\mathrm{NRA}(X)$ the subset of $L(X)$ consisting of the numerical radius-attaining operators.

Berg and Sims [**1**] have proved the "Bishop-Phelps type" result that $\mathrm{NRA}(X)$ is dense in $L(X)$ when X is uniformly convex. Elsewhere we have shown the same to be so for X being c_0, l_1, $L_1(\mu)$ or a uniformly smooth space.

In this note we consider the case of $X = C(K)$, the space of continuous real-valued functions on the compact Hausdorff space K. Following the lead of Johnson and Wolfe [**3**], we again show that $\mathrm{NRA}(C(K))$ is dense in $L(C(K))$.

We still do not know of any X for which $\mathrm{NRA}(X)$ is not dense in $L(X)$. It may be that Lindenstrauss's example, using a renorming of c_0, for which the norm-attaining operators are not dense in $L(X)$ [**4**] also serves in the present setup, but we have not yet found that to be so.

We introduce initially some definitions and notations.

We define the numerical radius of a bounded linear operator $T: X \to X$, denoted by $v(T)$, by

$$v(T) = \sup\{|x^*(Tx)|: (x, x^*) \in \Pi(X)\},$$

where $\Pi(X) = \{(x, x^*) \in X \times X^*: \|x^*\| = \|x\| = x^*(x) = 1\}$.

We say that T attains its numerical radius if there is $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^*(Tx_0)|$, and we denote the set of numerical radius-attaining operators by $\mathrm{NRA}(X)$.

If K is a compact Hausdorff space and X is a Banach space, we denote by $C_{\mathrm{w}^*}(K, X^*)$ the Banach space of continuous functions $F: K \to X^*$, where X^* is equipped with its w*-topology, with the norm $\|F\| = \sup\{\|F(t)\|: t \in K\}$.

It is a well-known result that $C_{\mathrm{w}^*}(K, X^*)$ can be identified, isomorphically and isometrically, with the space $L(X, C(K))$ of all bounded linear operators from X into $C(K)$, the identification being given by

$$(Tx)(t) = F(t)(x), \quad \forall\, t \in K, \forall\, x \in X,$$

where $T \in L(X, C(K))$ [**2**, p. 490].

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$M(K)$ denotes the space of regular Borel measures on K, with the norm of the variation, and is identified with $C(K)^*$.

In our case we will use the identification of $L(C(K))$ with $C_{w^*}(K, M(K))$.

For the proof of the result announced in the abstract we need several lemmas.

LEMMA 1. *Given $F \in C_{w^*}(K, M(K))$, $\varepsilon > 0$, $f \in C(K)$, $t_0 \in K$ and an open set $V \subset K$, there is U, an open neighborhood of t_0, such that*

(i) $|F(t)|(V) \geq |F(t_0)|(V) - \varepsilon, \forall t \in U$;

(ii) $F(t)(f) \geq F(t_0)(f) - \varepsilon, \forall t \in U$.

PROOF. First we show that the function $\nu \in M(K) \mapsto |\nu|(V) \in \mathbf{R}$ is lower semicontinuous, where $M(K)$ has its w*-topology.

In fact, if $\nu_0 \in M(K)$, by Hahn decomposition and regularity of ν_0 we can choose disjoint compact sets K^+ and K^-, contained in V, such that $|\nu_0|(K^+) = \nu_0(K^+)$, $|\nu_0|(K^-) = -\nu_0(K^-)$ and $|\nu_0|(V \setminus K^+ \cup K^-) < \varepsilon/3$.

Since K is compact Hausdorff, we can choose $f_0 \in C(K)$ with $|f_0(t)| \leq 1$, $\forall t \in K, f_0|_{K^+} = 1, f_0|_{K^-} = -1$ and $f_0|_{K \setminus V} = 0$.

Let $A = \{\nu \in M(K)\colon |\nu(f_0) - \nu_0(f_0)| < \varepsilon/3\}$. Then A is a w*-neighborhood of ν_0, and if $\nu \in A$ we have

$$\begin{aligned}
|\nu|(V) &\geq \int_V f_0\, d|\nu| \geq \left| \int_V f_0\, d\nu \right| = |\nu(f_0)| > |\nu_0(f_0)| - \frac{\varepsilon}{3} \\
&= \left| \int_V f_0\, d\nu_0 \right| - \frac{\varepsilon}{3} \geq \int_V f_0\, d\nu_0 - \frac{\varepsilon}{3} \\
&= \int_{K^+} d\nu_0 - \int_{K^-} d\nu_0 + \int_{V \setminus K^+ \cup K^-} f_0\, d\nu_0 - \frac{\varepsilon}{3} \\
&> \nu_0(K^+) - \nu_0(K^-) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} > |\nu_0|(V) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = |\nu_0|(V) - \varepsilon.
\end{aligned}$$

Since $F \in C_{w^*}(K, M(K))$, the composite function $t \in K \mapsto |F(t)|(V)$ is also lower semicontinuous. Thus there is an open neighborhood U_1 of t_0 such that for $t \in U_1$ we have $|F(t)|(V) > |F(t_0)|(V) - \varepsilon$.

Also, given $B = \{\nu \in M(K)\colon |\nu(f) - F(t_0)(f)| < \varepsilon\}$, which is a w*-neighborhood of $F(t_0) \in M(K)$, there is U_2, an open neighborhood of t_0 such that for $t \in U_2$ we have $F(t) \in B$, since $F \in C_{w^*}(K, M(K))$. Then if $t \in U_2$ we have $F(t)(f) \geq F(t_0)(f) - \varepsilon$.

Letting $U = U_1 \cap U_2$ we have that U is an open neighborhood of t_0 and, for $t \in U$, (i) and (ii) hold.

LEMMA 2. *Given $F \in C_{w^*}(K, M(K))$ and $\varepsilon > 0$, there are $f_0 \in C(K)$, $\|f_0\|_\infty = 1$ and $t_0 \in K$ such that $F(t_0)(f_0) > \|F\| - \varepsilon$ and $|f_0(t_0)| = 1$.*

PROOF. Let $t_0 \in K$ be such that $|F(t_0)|(K) > \|F\| - \varepsilon/3$.

For simplicity let us set $\mu_0 = F(t_0)$. Then $|\mu_0|(K) > \|F\| - \varepsilon/3$.

Using Hahn decomposition and regularity of μ_0, we can choose disjoint compact sets K^+ and K^- such that $|\mu_0|(K^+) = \mu_0(K^+)$, $|\mu_0|(K^-) = -\mu(K^-)$ and

$$|\mu_0|(K \setminus K^+ \cup K^-) < \varepsilon/3.$$

Then $\mu_0(K^+) - \mu_0(K^-) > |\mu_0|(K) - \varepsilon/3$.

Case I. $t_0 \in K^+ \cup K^-$.

Since K is compact Hausdorff, we can choose $f_0 \in C(K)$, $|f_0(t)| \leqslant 1$, $\forall\, t \in K$, $f_0|_{K^+} = 1$, and $f_0|_{K^-} = -1$.

Then

$$F(t_0)(f_0) = \int_K f_0\, d\mu_0 = \int_{K^+} d\mu_0 - \int_{K^-} d\mu_0 + \int_{K \setminus K^+ \cup K^-} f_0\, d\mu_0$$
$$= \mu_0(K^+) - \mu_0(K^-) + \int_{K \setminus K^+ \cup K^-} f_0\, d\mu_0.$$

Since

$$\left| \int_{K \setminus K^+ \cup K^-} f_0\, d\mu_0 \right| \leqslant |\mu_0|(K \setminus K^+ \cup K^-) < \varepsilon/3,$$

we get

$$F(t_0)(f_0) > \mu_0(K^+) - \mu_0(K^-) - \varepsilon/3 > |\mu_0|(K) - \varepsilon/3 - \varepsilon/3$$
$$> \|F\| - \varepsilon/3 - \varepsilon/3 - \varepsilon/3 = \|F\| - \varepsilon.$$

Obviously in this case we have $|f_0(t_0)| = 1$, since $t_0 \in K^+ \cup K^-$ and $|f_0|\,|_{K^+ \cup K^-} = 1$.

Case II. $t_0 \notin K^+ \cup K^-$.

Since $K^+ \cup \{t_0\}$ and K^- are again disjoint compact sets, let $f_0 \in C(K)$ be such that $|f_0(t)| \leqslant 1$, $\forall\, t \in K$, $f_0|_{K^+ \cup \{t_0\}} = 1$ and $f_0|_{K^-} = -1$.

As in Case I we have $F(t_0)(f_0) > \|F\| - \varepsilon$ and $f_0(t_0) = 1$, by definition of f_0.

As an easy consequence we have

COROLLARY 3. $v(T) = \|T\|$, $\forall\, T \in C(K)$.

The next lemma is a modification of a result of Johnson and Wolfe **[3]**.

LEMMA 4. *Given $F \in C_{w^*}(K, M(K))$ and $\varepsilon > 0$, there are open subsets V_1 and V_2 of K, with $\overline{V}_1 \cap \overline{V}_2 = \varnothing$, $V_2 \neq \varnothing$, and there are $f_1 \in C(K)$, $\|f_1\|_\infty = 1$, and $F_1 \in C_{w^*}(K, M(K))$ such that*

(i) $|f_1(t)| = 1$, $\forall\, t \in K \setminus V_1$;
(ii) $|F_1(t)|(V_1) = 0$, $\forall\, t \in V_2$;
(iii) $F_1(t)(f_1) > \|F_1\| - \varepsilon$, $\forall\, t \in V_2$;
(iv) $\|F - F_1\| < \varepsilon$.

PROOF. Let $t_0 \in K$ be such that $|F(t_0)|(K) > \|F\| - \varepsilon/4$.

Using (B^+, B^-) a Hahn decomposition of K for $\mu_0 = F(t_0)$ and the regularity of μ_0, choose $K^+ \subset B^+$ and $K^- \subset B^-$ compact sets such that

$$\mu_0(K^+) - \mu_0(K^-) > |\mu_0|(K) - \varepsilon/4 > \|F\| - \varepsilon/2.$$

As in the proof of Lemma 2, let $f_0 \in C(K)$, $\|f_0\|_\infty = 1$, be such that $f_0|_{K^+} = 1$, $f_0|_{K^-} = -1$ and $|f_0|\,|_{K^+ \cup K^- \cup \{t_0\}} = 1$.

For each $\alpha \in]0, 1[$, let $A_\alpha = \{t \in K: |f_0(t)| < \alpha\}$.

Case I. $A_\alpha = \varnothing$, $\forall\, \alpha \in]0, 1[$.

In this case, $|f_0(t)| = 1$, $\forall\, t \in K$. Define $f_1 = f_0$, $V_1 = \varnothing$ and $F_1 = F$. Then (i) and (ii) hold for $t \in K$ and (iv) also is satisfied.

Moreover,

$$F_1(t_0)(f_1) = F(t_0)(f_0) \geqslant \mu_0(K^+) - \mu_0(K^-) - \varepsilon/4 > \|F\| - 3\varepsilon/4.$$

By Lemma 1, using $\varepsilon/4$, there is $V_2 \subset K$, an open neighborhood of t_0, such that $F_1(t)(f_1) \geqslant F_1(t_0)(f_1) - \varepsilon/4, \forall\, t \in V_2$.

Then $F_1(t)(f_1) \geqslant \|F\| - \varepsilon = \|F_1\| - \varepsilon, \forall\, t \in V_2$ and (iii) holds.

Obviously, $V_2 \neq \varnothing$ and $\overline{V}_1 \cap \overline{V}_2 = \varnothing$, and we are done.

Case II. There is $\alpha_0 \in]0,1[$ with $A_{\alpha_0} \neq \varnothing$.

In this case let β_0 be such that $\alpha_0 < \beta_0 < 1$.

Define $V_1 = \{t \in V\colon |f_0(t)| < \alpha_0\} = A_{\alpha_0}$ and $W = \{t \in K\colon |f_0(t)| > \beta_0\}$. Then V_1 and W are open sets, $\overline{V}_1 \cap W = \varnothing$ and $\{t_0\} \cup K^+ \cup K^- \subset W$.

Since $A_{\alpha_0} \neq \varnothing$, fix $t_1 \in V_1$ and choose $f_1, g \in C(K)$, $|f_1(t)| \leqslant 1$, $0 \leqslant g(t) \leqslant 1$, $\forall\, t \in K$, such that

$$f_1(t) = \begin{cases} 1 & \text{if } t \in \overline{(K \setminus V_1) \cap B^+}\,, \\ -1 & \text{if } t \in \overline{(K \setminus V_1) \cap B^-}\,, \\ 0 & \text{if } t = t_1 \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} 0 & \text{if } t \in \overline{W}, \\ 1 & \text{if } t \in \overline{V}_1. \end{cases}$$

Then (i) holds and since

$$[(1-g)f_1](t) = 1 \quad \text{if } t \in K^+, \qquad [(1-g)f_1](t) = -1 \quad \text{if } t \in K^-$$

and

$$|[(1-g)f_1](t)| \leqslant 1 \quad \text{if } t \in (B^+ \setminus K^+) \cup (B^- \setminus K^-),$$

we have

$$\begin{aligned} F(t_0)((1-g)f_1) &= \int_K (1-g)f_1\,d\mu_0 \\ &= \int_{K^+} (1-g)f_1\,d\mu_0 + \int_{K^-} (1-g)f_1\,d\mu_0 \\ &\quad + \int_{(B^+\setminus K^+)\cup(B^-\setminus K^-)} (1-g)f_1\,d\mu_0 \\ &\geqslant \mu_0(K^+) - \mu_0(K^-) - |\mu_0|((B^+ \setminus K^+) \cup (B^- \setminus K^-)) \\ &\geqslant |\mu_0|(K) - \varepsilon/4 - \varepsilon/4 > \|F\| - 3\varepsilon/4. \end{aligned}$$

By Lemma 1, using $\varepsilon/4$, there is $U \subset K$ an open neighborhood of t_0 such that for each $t \in U$,

$$F(t)((1-g)f_1) \geqslant F(t_0)((1-g)f_1) - \varepsilon/4 > \|F\| - \varepsilon$$

and

$$|F(t)|(W) \geqslant |F(t_0)|(W) - \varepsilon/4 > \|F\| - \varepsilon.$$

We can take $U \cap \overline{V}_1 = \varnothing$, since $t_0 \notin \overline{V}_1$. Let $V_2 \subset U$ be an open set such that $t_0 \in V_2$ and $\overline{V}_2 \subset U$. In particular, $V_2 \neq \varnothing$ and $\overline{V}_1 \cap \overline{V}_2 = \varnothing$.

Choose $h \in C(K)$, $\|h\|_\infty = 1$, $h(t) = 1$ if $t \in \overline{V}_2$ and $h(t) = 0$ if $t \in K \setminus U$ and define $F_1\colon K \to M(K)$ by $F_1(t) = [1 - h(t)g]F(t), \forall\, t \in K$, which means

$$F_1(t)(p) = F(t)([1 - h(t)g]\,p), \qquad \forall\, p \in C(K).$$

Since $g \in C(K)$, $F_1(t) \in M(K)$, $\forall\, t \in K$ and since $h \in C(K)$ and $F \in C_{w*}(K, M(K))$, $F_1 \in C_{w*}(K, M(K))$. Also $|F_1(t)|(K) \leqslant |F(t)|(K)$, since $\|1 - h(t)g\|_\infty \leqslant 1, \forall\, t \in K$, and then $\|F_1\| \leqslant \|F\|$.

If $t \in V_2$, $h(t) = 1$ and $F_1(t) = (1 - g)F(t)$. Since $g|_{\overline{V}_1} = 1$, $|F_1(t)|(V_1) = 0$ and (ii) holds. Also

$$F_1(t)(f_1) = F(t)((1 - g)f_1) > \|F\| - \varepsilon \geqslant \|F_1\| - \varepsilon$$

and (iii) holds. For (iv), note that

$$|F(t) - F_1(t)|(K) = |h(t)gF(t)|(K) = 0 \quad \text{if } t \in K \setminus U,$$

since $h|_{K\setminus U} = 0$ and

$$|F(t) - F_1(t)|(K) \leqslant |gF(t)|(K) \quad \text{if } t \in U.$$

But $g|_{\overline{W}} = 0$ and then

$$\begin{aligned} |gF(t)|(K) &= |gF(t)|(K \setminus \overline{W}) \leqslant |F(t)|(K \setminus \overline{W}) \\ &= |F(t)|(K) - |F(t)|(\overline{W}) \leqslant \|F\| - |F(t)|(W) \\ &< \|F\| - (\|F\| - \varepsilon) = \varepsilon \quad \text{if } t \in U. \end{aligned}$$

Then

$$\|F - F_1\| = \sup\{|F(t) - F_1(t)|(K) : t \in K\} \leqslant \varepsilon.$$

The proof of the next lemma can be found in [**3**, Lemma 2.4].

LEMMA 5. *Let $F \in C_{w*}(K, M(K))$, $V_1, V_2 \subset K$ open sets, $t_0 \in V_2$, $f_0 \in C(K)$, $\|f_0\|_\infty = 1$, be such that*

(a) $|F(t)|(V_1) = 0, \forall\, t \in V_2$;

(b) $F(t_0)(f_0) \geqslant \|F\| - \varepsilon$;

(c) $|f_0(t)| = 1, \forall\, t \in K \setminus V_1$.

Then for every $r > 2/3$, there is $F_1 \in C_{w*}(K, M(K))$, and there is $t_1 \in V_2$ such that

(i) $|F_1(t)|(V_1) = 0, \forall\, t \in V_2$;

(ii) $F_1(t_1)(f_0) \geqslant \|F_1\| - r\varepsilon$;

(iii) $\|F - F_1\| < r\varepsilon$.

THEOREM 6. $\overline{\mathrm{NRA}(C(K))} = L(C(K))$.

PROOF. Let $T \in L(C(K))$ and $\varepsilon > 0$ be given, and let $F \in C_{w*}(K, M(K))$ be the representative of T.

Take $2/3 < r < 1$ and apply Lemma 4 to get $F_0 \in C_{w*}(K, M(K))$, $V_1, V_2 \subset K$ open sets, $\overline{V}_1 \cap \overline{V}_2 = \varnothing$, $V_2 \neq \varnothing$, $f_0 \in C(K)$, $\|f_0\|_\infty = 1$, such that

(a) $|F_0(t)|(V_1) = 0, \forall\, t \in V_2$;

(b) $F_0(t)(f_0) > \|F_0\| - \varepsilon(1 - r), \forall\, t \in V_2$;

(c) $|f_0(t)| = 1, \forall\, t \in K \setminus V_1$;

(d) $\|F - F_0\| < \varepsilon(1 - r)$.

Choose $t_0 \in V_2$ such that

(b′) $F_0(t_0)(f_0) > \|F_0\| - \varepsilon(1 - r)$, and let $\lambda = \|F_0\| - F_0(t_0)(f_0)$. Then $0 \leqslant \lambda < \varepsilon(1 - r)$.

Case I. $\lambda = 0$.

In this case, $\|F_0\| = F_0(t_0)(f_0) = \delta_{t_0}(T_0 f_0)$, where $T_0 \in L(C(K))$ corresponds to F_0.

Defining $\mu_0 = (\operatorname{sgn} f_0(t_0))\delta_{t_0}$, we have $\mu_0(f_0) = |f_0(t_0)| = 1$, since $t_0 \in V_2$ and $V_1 \cap V_2 = \varnothing$, and $|\mu_0|(K) = 1$. Then $(f_0, \mu_0) \in \Pi(C(K))$.

Also we have $T_0 \in \mathrm{NRA}(C(K))$, for

$$|\mu_0(T_0 f_0)| = |\delta_{t_0}(T_0 f_0)| = \|F_0\| = \|T_0\|.$$

From (d), $\|T - T_0\| = \|F - F_0\| < \varepsilon(1 - r) < \varepsilon$, and we are done.

Case II. $\lambda > 0$.

By definition of λ,

(b″) $F_0(t_0)(f_0) = \|F_0\| - \lambda$.

Now (a), (b″) and (c) allow us to apply Lemma 5 and get $F_1 \in C_{w^*}(K, M(K))$ and $t_1 \in V_2$ such that

(a_1) $|F_1(t)|(V_1) = 0, \forall t \in V_2$;

(b_1) $F_1(t_1)(f_0) \geqslant \|F_1\| - r\lambda$;

(d_1) $\|F_0 - F_1\| < r\lambda$.

Again (a_1), (b_1) and (c) allow us to apply Lemma 5 and get $F_2 \in C_{w^*}(K, M(K))$ and $t_2 \in V_2$ such that

(a_2) $|F_2(t)|(V_1) = 0, \forall t \in V_2$;

(b_2) $F_2(t_2)(f_0) \geqslant \|F_2\| - r^2\lambda$;

(d_2) $\|F_1 - F_2\| < r^2\lambda$.

Following in this way we get sequences $\{F_n\}$ in $C_{w^*}(K, M(K))$ and $\{t_n\}$ in V_2 such that for each $n \in \mathbf{N}$,

(b_n) $F_n(t_n)(f_0) \geqslant \|F_n\| - r^n\lambda$;

(d_n) $\|F_{n-1} - F_n\| \leqslant r^n\lambda$.

Since K is compact, $\{t_n\}$ has a subsequence convergent to some $\tilde{t} \in K$. But $t_n \in V_2, \forall n \in \mathbf{N}$ and then $\tilde{t} \in \overline{V}_2$. We still denote this subsequence by $\{t_n\}$.

On the other hand, if $m > n \geqslant 1$, by (d_n) it follows that

$$\|F_n - F_m\| \leqslant \sum_{k=n+1}^{m} \|F_k - F_{k-1}\| \leqslant \left(\sum_{k=n+1}^{m} r^k\right)\lambda.$$

Since $r < 1$, this shows that $\{F_n\}$ is Cauchy in $C_{w^*}(K, M(K))$ and so is $\{T_n\}$ in $L(C(K))$, where T_n corresponds to F_n, $\forall n \in \mathbf{N}$.

Let $\tilde{T} \in L(C(K))$ be the limit of $\{T_n\}$ and $\tilde{F}$ its correspondent in $C_{w^*}(K, M(K))$.

We have

$$\begin{aligned}\|T - T_n\| &\leqslant \|T - T_0\| + \|T_0 - T_n\| \leqslant \varepsilon(1 - r) + \left(\sum_{k=1}^{n} r^k\right)\lambda \\ &\leqslant \varepsilon(1 - r) + \frac{r}{1 - r}\varepsilon(1 - r) = \varepsilon, \qquad \forall n \in \mathbf{N}.\end{aligned}$$

From this it follows that $\|T - \tilde{T}\| \leqslant \varepsilon$.

It remains to show that $\tilde{T} \in \mathrm{NRA}(C(K))$.

From $|\tilde{F}(t_n)(f_0) - F_n(t_n)(f_0)| \leqslant \|\tilde{F} - F_n\|$ and (b_n) we get

$$\begin{aligned}\tilde{F}(t_n)(f_0) &\geqslant F_n(t_n)(f_0) - \|\tilde{F} - F_n\| \geqslant \|F_n\| - r^n\lambda - \|\tilde{F} - F_n\| \\ &\geqslant \|\tilde{F}\| - r^n\lambda - 2\|\tilde{F} - F_n\|, \qquad \forall\, n \in \mathbf{N},\end{aligned}$$

and since

$$\|\tilde{F} - F_n\| \leqslant \left(\sum_{k=n+1}^{\infty} r^k\right)\lambda = \frac{r^{n+1}}{1-r}\lambda < r^{n+1}, \qquad \forall\, n \in \mathbf{N},$$

we have

$$\tilde{F}(t_n)(f_0) \geqslant \|\tilde{F}\| - r^n\varepsilon(1-r) - 2r^{n+1}\varepsilon = \|\tilde{F}\| - r^n\varepsilon(3-r).$$

Now, since $\tilde{F}$ is w*-continuous and $t_n \to \tilde{t}$ and $r^n \to 0$, we have

$$\tilde{F}(\tilde{t})(f_0) \geqslant \|\tilde{F}\| \quad \text{or} \quad \delta_{\tilde{t}}(\tilde{T}f_0) \geqslant \|\tilde{T}\|.$$

Also $|f_0(\tilde{t})| = \lim_{n\to\infty} |f_0(t_n)| = 1$, since f_0 is continuous, $t_n \to \tilde{t}$ and $|f_0(t_n)| = 1$, because $t_n \in V_2$ and $\overline{V}_2 \cap \overline{V}_1 = \varnothing$.

Defining $\tilde{\mu} = (\operatorname{sgn} f_0(\tilde{t}))\delta_{\tilde{t}}$, we have $\tilde{\mu} \in M(K)$, $|\tilde{\mu}|(K) = 1$, and $\tilde{\mu}(f_0) = |f_0(\tilde{t})| = 1$. Then $(f_0, \tilde{\mu}) \in \Pi(C(K))$.

Since $|\tilde{\mu}(\tilde{T}f_0)| = |\delta_{\tilde{t}}(\tilde{T}f_0)| \geqslant \|\tilde{T}\|$ and $\|\tilde{T}\| = v(\tilde{T})$, by Corollary 3, we get $|\tilde{\mu}(\tilde{T}f_0)| \geqslant v(\tilde{T})$. But $v(\tilde{T}) \geqslant |\tilde{\mu}(\tilde{T}f_0)|$, and then $v(\tilde{T}) = |\tilde{\mu}(\tilde{T}f_0)|$ and $\tilde{T} \in \mathrm{NRA}(C(K))$.

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