PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 4, December 1985

CLASSIFICATION OF SEMICROSSED PRODUCTS OF FINITE-DIMENSIONAL C^*-ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}, \mathfrak{B}$ be finite-dimensional C^*-algebras with automorphisms α, β, respectively. Then the semicrossed products $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$, $\mathbf{Z}^+ \times_\beta \mathfrak{B}$ are isomorphic iff there is an isomorphism $\psi\colon \mathfrak{A} \to \mathfrak{B}$ and a unitary $U \in \mathfrak{B}$ such that $\beta \circ \psi = (\operatorname{Ad} U)\psi \circ \alpha$.

I. Introduction. In 1969 Arveson and Josephson [**1**] studied a class of operator algebras which arise in the following way: Let X be a locally compact Hausdorff space equipped with a self-homeomorphism ϕ, and an invariant regular Borel probability measure μ. Let L be the representation of the continuous functions vanishing at infinity, $C_0(X)$, on $L^2(X,\mu)$ given by $L_f g = fg$ ($f \in C_0(X)$, $g \in L^2(X,\mu)$) and U the unitary on $L^2(X,\mu)$ coming from the action of $\phi\colon U_g = g \circ \phi$. It is readily verified that the collection of all finite sums of the form $\sum_{i=1}^n L_{f_i} U^i$ ($f_i \in C_0(X)$, $1 \le i \le n, n \in \mathbf{Z}^+$) is an algebra, and the Arveson-Josephson algebra $\mathfrak{A}(X,\phi)$ is the norm closure in $\mathcal{B}(L^2(X,\mu))$ of this set of operators. The pairs (X_1,ϕ_1) and (X_2,ϕ_2) are said to be conjugate if there exists a homeomorphism $\psi\colon X_2 \to X_1$ such that $\phi_1 \circ \psi = \psi \circ \phi_2$. The principal aim of [**1**] was to show, under certain conditions, that the algebras $\mathfrak{A}(X_1,\phi_1)$ and $\mathfrak{A}(X_2,\phi_2)$ are isomorphic if and only if the pairs (X_1,ϕ_1), (X_2,ϕ_2) are conjugate.

In [**8**] it was shown that the Arveson-Josephson algebras can be considered as an example of a larger class of algebras called semicrossed products. Given a C^*-algebra $\mathfrak{A}$ and an endomorphism α of $\mathfrak{A}$, the semicrossed product $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is an operator algebra which can be defined by a universal mapping property. This definition of the semicrossed product is analogous to that of the crossed product and will be discussed below. If $\mathfrak{A} = C_0(X)$, where X is a locally compact Hausdorff space, $\alpha(f) = f \circ \phi$, $f \in C_0(X)$ and ϕ a self-homeomorphism of X (satisfying [**1**, conditions 1.1(i), (ii) and (iii)]), then $\mathfrak{A}(X,\phi) = \mathbf{Z}^+ \times_\alpha C_0(X)$. Thus, the question of Arveson and Josephson, as to whether the isomorphism of the algebras $\mathfrak{A}(X_1,\phi_1)$, $\mathfrak{A}(X_2,\phi_2)$ implies the conjugacy of the pairs (X_1,ϕ_1), (X_2,ϕ_2), can be placed in the more general setting of semicrossed products. In this paper we show that if $\mathfrak{A}$ and $\mathfrak{B}$ are finite-dimensional C^*-algebras with automorphisms α and β, respectively, then the semicrossed products $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ and $\mathbf{Z}^+ \times_\beta \mathfrak{B}$ are isomorphic if and only if there is an isomorphism $\psi\colon \mathfrak{A} \to \mathfrak{B}$, and a unitary $U \in \mathfrak{B}$, such that $\beta \circ \psi = (\operatorname{Ad} U)\psi \circ \alpha$. This result emerges from a detailed analysis of the Banach algebras $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ and the strong structure space of such algebras, where $\mathfrak{A}$ is a finite-dimensional C^*-algebra. The analogous result for C^*-crossed products fails. It is possible for $\mathbf{Z} \times_\alpha \mathfrak{A}$, $\mathbf{Z} \times_\beta \mathfrak{B}$ to be isomorphic without $\mathfrak{A}, \mathfrak{B}$ being isomorphic.

Received by the editors December 3, 1984 and, in revised form, February 1, 1985. Presented by the second author of the 91st Annual Meeting of the AMS in January, 1985.

1980 *Mathematics Subject Classification.* Primary 47D25, 46H20.

We begin with a discussion of representation theory for semicrossed products and then restrict our attention to semicrossed products of finite-dimensional C^*-algebras.

II. Representations of semicrossed products. Let α be an automorphism of a C^*-algebra $\mathfrak{A}$ (all C^*-algebras will be assumed separable). Recall [**7**] that the Banach algebra $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$ consists of functions $F: \mathbf{Z} \to \mathfrak{A}$ such that $\|F\|_1 \equiv \sum_{n \in \mathbf{Z}} \|F(n)\| < \infty$, and multiplication $(FG)(m) = \sum_{n \in \mathbf{Z}} F(n)\alpha^n(G(m-n))$, $F, G \in l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$. Also, $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$ has an involution given by

$$F^*(n) = \alpha^n(F(-n)^*).$$

The C^*-crossed product, $\mathbf{Z} \times_\alpha \mathfrak{A}$, is then defined as the C^*-enveloping algebra of the involutive Banach algebra $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$. However, if we ignore the involution in $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$, the crossed product $\mathbf{Z} \times_\alpha \mathfrak{A}$ can be realized as the enveloping Banach algebra with respect to the class of contractive Hilbert space representations. Specifically, if we define a norm on $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$ by

$$\|F\| = \sup_{\Pi \in \mathcal{C}} \|\Pi(F)\|,$$

where $\Pi \in \mathcal{C}$ if and only if Π is a contractive representation $\Pi: l^1(\mathbf{Z}, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$, for some (separable) Hilbert space $\mathcal{H}$, then this norm agrees with the crossed product norm of F. While these facts, which follow from [**5**, Theorem 4.4], will not be needed later, they serve as motivation for the definition of the semicrossed product.

II.1. Let $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ be the subalgebra of $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$ consisting of functions supported on the nonnegative integers. Note that $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ is not closed under the involution in $l^1(\mathbf{Z}, \mathfrak{A}, \alpha)$. The semicrossed product $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is then defined as the enveloping Banach algebra with respect to the class of contractive Hilbert space representations. In other words, by analogy with the preceding paragraph, if $\mathcal{C}$ is the class of contractive representations $\Pi: l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$, for some (separable) Hilbert space $\mathcal{H}$, then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is the completion of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ with respect to the norm

$$\|F\| = \sup_{\Pi \in \mathcal{C}} \|\Pi(F)\|.$$

II.2. DEFINITION. Let $\rho: \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a representation of the C^*-algebra $\mathfrak{A}$, and let $T \in \mathcal{B}(\mathcal{H})$, a contraction (resp., isometry). We say that the pair (T, ρ) is a contractive (resp., isometric) covariant representation of the pair $(\mathfrak{A}, \alpha)$ if $T\rho(x) = \rho(\alpha(x))T$ holds for all $x \in \mathfrak{A}$.

Observe that, if $\mathfrak{A}$ is unital, there is a one-to-one correspondence between contractive representations of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ and contractive covariant representations of $(\mathfrak{A}, \alpha)$ given by $\Pi \leftrightarrow (T, \rho)$ if and only if $\rho(x) = \Pi(\delta_0 \otimes x)$, and $T = \Pi(\delta_1 \otimes 1)$. (If $\mathfrak{A}$ is nonunital, T can be obtained by passing to the multiplier algebra.) Note that ρ is a contractive representation of $\mathfrak{A}$, hence a *-representation. If Π corresponds to (T, ρ), we will write $\Pi = T \times \rho$. Then

$$(T \times \rho)\left(\sum_{n \geq 0} \delta_n \otimes x_n\right) = \sum_{n \geq 0} \rho(x_n)T^n.$$

Obviously, contractive representations of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ can be extended to representations of the enveloping algebra $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$. We will not make a notational distinction between representations of $l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$ and those of $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$.

II.3. PROPOSITION. *Let $\mathcal{CI}$ be the class of all isometric covariant representations of $(\mathfrak{A}, \alpha)$. If $F \in \mathbf{Z}^+ \times_\alpha \mathfrak{A}$, then*

$$\|F\| = \sup_{(V,\rho)\in \mathcal{CI}} \|(V \times \rho)(F)\|.$$

PROOF. Since $\mathcal{CI}$ is a subclass of the class of all contractive covariant representations, the right-hand side is clearly less than or equal to the left-hand side. To complete the proof, it is enough to show that any contractive covariant representation (T, ρ) of $(\mathfrak{A}, \alpha)$ can be dilated to an isometric covariant representation.

Thus, let (T, ρ) be a contractive covariant representation of $(\mathfrak{A}, \alpha)$ on a Hilbert space $\mathcal{H}$, and let V be the minimal isometric dilation of T, as in [**6**], on the Hilbert space $H^2(\mathcal{H})$ consisting of sequences $(\xi_0, \xi_1, \ldots)$, $\xi_n \in \mathcal{H}$, $\sum_{n\geq 0} \|\xi_n\|^2 < \infty$. V is given by $V(\xi_0, \xi_1, \ldots) = (T\xi_0, D_T\xi_0, \xi_1, \ldots)$, where $D_T = (I_{\mathcal{H}} - T^*T)^{1/2}$. Define the representation $\tilde{\rho}$ of $\mathfrak{A}$ on $H^2(\mathcal{H})$ by

$$\tilde{\rho}(x)(\xi_0, \xi_1, \xi_2, \ldots) = (\rho(x)\xi_0, \rho(\alpha^{-1}x)\xi_1, \rho(\alpha^{-2}x)\xi_2, \ldots).$$

We claim that $(V, \tilde{\rho})$ is an isometric covariant representation of $(\mathfrak{A}, \alpha)$. Now $\rho(\alpha x)T = T\rho(x)$ for all $x \in \mathfrak{A}$ implies $T^*\rho(\alpha x) = \rho(x)T^*$. So $\rho(x)T^*T = T^*\rho(\alpha x)T = T^*T\rho(x)$, and hence $\rho(x)$ commutes with D_T^2 and D_T as well. We compute

$$\begin{aligned}
\tilde{\rho}(\alpha x)(\xi_0, \xi_1, \xi_2, \ldots) &= \tilde{\rho}(\alpha x)(T\xi_0, D_T\xi_0, \xi_1, \xi_2, \ldots) \\
&= (\rho(\alpha x)T\xi_0, \rho(x)D_T\xi_0, \rho(\alpha^{-1}x)\xi_1, \ldots) \\
&= (T\rho(x)\xi_0, D_T\rho(x)\xi_0, \rho(\alpha^{-1}x)\xi_1, \ldots) \\
&= V(\rho(x)\xi_0, \rho(\alpha^{-1}x)\xi_1, \ldots) = V\tilde{\rho}(x)(\xi_0, \xi_1, \ldots).
\end{aligned}$$

This proves the claim. Thus $(V, \tilde{\rho})$ is the desired dilation of (T, ρ). □

II.4. COROLLARY. *Writing the unilateral shift on $H^2(\mathcal{H})$ as multiplication by z, M_z, we conclude $\|F\| = \sup \|M_z \times \tilde{\rho}(F)\|$, $F \in \mathbf{Z}^+ \times_\alpha \mathfrak{A}$, where the supremum is taken over all representations $(\rho, \mathcal{H})$ of $\mathfrak{A}$.*

II.5. *Notation.* If ρ is a representation of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ we will denote by $\tilde{\rho}$ the representation of $\mathfrak{A}$ on $H^2(\mathcal{H})$ as constructed in the above proof.

II.6. Let S be a locally compact Hausdorff space, and $\phi: S \to S$ a homeomorphism. Let $\mathfrak{A}$ be the C^*-algebra of continuous functions from S into $M_N(\mathbf{C})$ (complex N by N matrices) which vanish at infinity, and α the automorphism of $\mathfrak{A}$ given by $\alpha(f) = f \circ \phi$. Denote by Π_s the representation $\Pi_s(f) = f(s)$, $f \in \mathfrak{A}$. We can consider Π_s $(s \in S)$ as acting on a fixed N-dimensional Hilbert space $\mathcal{H}_N$.

PROPOSITION. *For $F \in \mathbf{Z}^+ \times_\alpha \mathfrak{A}$, $\|F\| = \sup_{s\in S} \|M_z \times \tilde{\Pi}_s(F)\|$.*

PROOF. This follows from [**8**, Propositions II.7 and II.8] in case $N = 1$, but with a suitable change of notation the same proof is, in fact, valid for any positive integer N.

III. Semicrossed products of finite-dimensional C^*-algebras. Let $A(\mathbf{D})$ denote the disk algebra, i.e., the commutative Banach algebra of continuous functions on the closed unit disk $\overline{\mathbf{D}}$, which are holomorphic in the interior. Fix a positive integer K, and let $\mathfrak{B}_K$ be the algebra of all K by K matrices of functions $[f_{ij}]_{0\leq i,j\leq K-1}$, $f_{ij} \in A(\mathbf{D})$ and of the form $f_{ij}(z) = \sum_{n\geq 0} a_n^{(ij)} z^{l+nK}$, where

$0 \le l \le K-1$ and $l = i - j \pmod K$. There are various (equivalent) norms under which $\mathcal{B}_K$ is a Banach algebra. We describe one such norm.

Let H^2 refer to the classical Hardy space of holomorphic functions $\xi(z) = \sum_{n\ge 0} \xi_n z^n$ in the unit disk having nontangential L^2 boundary values with inner product

$$(\xi,\eta) = \frac{1}{2\pi}\int_{|z|=1} \xi(z)\overline{\eta(z)}\,dz \qquad (\xi,\eta\in H^2).$$

Let H_j^2 $(0 \le j \le K-1)$ denote the subspace of functions of the form $\xi(z) = \sum_{n\ge 0} \xi_{j+nK} z^{j+nK}$. The subspaces H_j^2, H_k^2 are orthogonal if $j \ne k, j,k \in \{0,1,2,\dots,K-1\}$. Now a matrix $[f_{ij}] \in \mathcal{B}_K$ maps a vector $[\xi^0(z),\xi^1(z),\dots,\xi^{K-1}(z)] \in \bigoplus_{j=0}^{K-1} H_j^2$ to a vector $[\eta^0(z),\eta^1(z),\dots,\eta^{K-1}(z)] \in \bigoplus_{j=0}^{K-1} H_j^2$ by means of

$$\eta^k(z) = \sum_{j=0}^{K-1} f_{kj}(z)\xi^j(z).$$

The norm of $\mathcal{B}_K$ is defined as the operator norm associated with this representation. Consider the algebra $\mathcal{B}_K \otimes M_N(\mathbf{C})$. An element in $\mathcal{B}_K \otimes M_N(\mathbf{C})$ may be considered as an operator on the Hilbert space $\left(\bigoplus_{j=0}^{K-1} H_j^2\right) \otimes \mathbf{C}^N$. Define the norm on $\mathcal{B}_K \otimes M_N(\mathbf{C})$ to be the operator norm associated with this representation. If $K > 1$ or $N > 1$, $\mathcal{B}_K \otimes M_N(\mathbf{C})$ is a noncommutative nonselfadjoint operator algebra. $\mathcal{B}_1 \otimes M_1(\mathbf{C}) = A(\mathbf{D})$ [**8**]. The maximal ideals of $\mathcal{B}_k$ are of codimension 1 or K^2. Those of codimension one are kernels of the representations $F = [f_{ij}] \to f_{kk}(0)$, for $0 \le k \le K-1$. Those of codimension K^2 are kernels of the representations $F = [f_{ij}] \to [f_{ij}(\lambda)]$, for $\lambda \in \overline{\mathbf{D}}$, $\lambda \ne 0$ [**8**].

III.1. LEMMA. *For every positive integers K and N, the maximal modular ideals in $\mathcal{B}_K \otimes M_N(\mathbf{C})$ are of the form $M \otimes M_N(\mathbf{C})$, where $M \subset \mathcal{B}_K$ is a maximal modular ideal.*

PROOF. Cf. [**2**]. □

III.2. PROPOSITION. *Let K and N be positive integers, and $\mathfrak{A} = KM_N(\mathbf{C})$ be K copies of $M_N(\mathbf{C})$. Let $\alpha \in \mathrm{Aut}(\mathfrak{A})$ be given by*

$$\alpha(A_0, A_1, \dots, A_{K-1}) = (A_{K-1}, A_0, \dots, A_{K-2}),$$

then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathcal{B}_K \otimes M_N(\mathbf{C})$.

PROOF. The proof for $N = 1$ is given in [**8**, III.2]. The general case is analogous. □

III.3. COROLLARY. *Let $\mathfrak{A} = KM_N(\mathbf{C})$ as above, $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that*

$$\alpha(A_0, A_1, \dots, A_{K-1}) = (A_{K-1}, A_0, \dots, A_{K-2});$$

then the semicrossed product $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is strongly semisimple.

III.4. PROPOSITION. *Let $\mathfrak{A}$ be a C^*-algebra with identity, $\alpha, \beta \in \mathrm{Aut}(\mathfrak{A})$ such that $\alpha(A) = \mathrm{Ad}(U)\beta(A)$, for some unitary $U \in \mathfrak{A}$. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathbf{Z}^+ \times_\beta \mathfrak{A}$.*

PROOF. Define $\psi\colon l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha) \to l^1(\mathbf{Z}^+, \mathfrak{A}, \beta)$ by

$$\psi\left(\sum_{n\geq 0} \delta_n^\alpha \otimes A_n\right) = \sum_{n\geq 0} \delta_n^\beta \otimes A_n \alpha^{n-1}(U)\cdots\alpha(U)U;$$

it is a straightforward though tedious computation to show than ψ is an isomorphism. Hence the enveloping algebras $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ and $\mathbf{Z}^+ \times_\beta \mathfrak{A}$ are isomorphic. □

III.5. PROPOSITION. *Let $\mathfrak{A} = KM_N(\mathbf{C})$, and $\alpha \in \mathrm{Aut}(\mathfrak{A})$ any automorphism which acts transitively on the factors. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathcal{B}_K \otimes M_N(\mathbf{C})$.*

PROOF. Let $\mathfrak{A} = \bigoplus_{j=0}^{K-1} \mathfrak{A}_j$, $\mathfrak{A}_j \cong M_N(\mathbf{C})$, $0 \leq j \leq K-1$. Without loss of generality we may arrange the factors so that $\alpha(\mathfrak{A}_j) = \mathfrak{A}_{j+1} \pmod K$. Let $\sigma\colon \mathfrak{A} \to \mathfrak{A}$ be the automorphism $\sigma(A_0, A_1, \ldots, A_{K-1}) = (A_{K-1}, A_0, \ldots, A_{K-2})$. Now $\sigma^{-1} \circ \alpha|\mathfrak{A}_j$ maps $\mathfrak{A}_j$ onto $\mathfrak{A}_j$, hence there exists a unitary $U_j \in \mathfrak{A}_j$ such that $\sigma^{-1} \circ \alpha|\mathfrak{A}_j(A_j) = \mathrm{Ad}(U_j)A_j$. Thus

$$\alpha(A_0, A_1, \ldots, A_{K-1}) = (\mathrm{Ad}(U_{K-1})A_{K-1}, \mathrm{Ad}(U_0)A_0, \ldots, \mathrm{Ad}(U_{K-2})A_{K-2}).$$

Therefore, if $U = \bigoplus_{j=0}^{K-1} U_j$, it follows that $\alpha(A) = \sigma(\mathrm{Ad}(U)A)$, $A \in \mathfrak{A}$. Hence $\alpha(A) = \mathrm{Ad}(W)\sigma(A)$, $W = \sigma(U)$. This implies that $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ by III.4. But by Proposition III.2 we have that $\mathbf{Z}^+ \times_\sigma \mathfrak{A}$ is isomorphic to $\mathcal{B}_K \otimes M_N(\mathbf{C})$. The result follows. □

III.6. COROLLARY. *Let $\mathfrak{A} = KM_N(\mathbf{C})$, and $\alpha \in \mathrm{Aut}(\mathfrak{A})$ any automorphism which acts transitively on the factors. Then the semicrossed product $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is strongly semisimple.*

III.7. PROPOSITION. *Let $\mathfrak{A}$ be an arbitrary finite-dimensional C^*-algebra, and $\alpha \in \mathrm{Aut}(\mathfrak{A})$. Decompose $\mathfrak{A} \cong \bigoplus_{j=1}^{r} \mathfrak{A}_j$, where each $\mathfrak{A}_j \cong K_j M_{N_j}(\mathbf{C})$, $\alpha(\mathfrak{A}_j) = \mathfrak{A}_j$, and α acts transitively on the direct summands of $\mathfrak{A}_j$. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\bigoplus_{j=1}^{r} \mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})$.*

PROOF. Let $\alpha_j = \alpha|\mathfrak{A}_j$, and $\sum_{n\geq 0} \delta_n \otimes_n A \in l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha)$, with

$${}_nA = ({}_nA_1, {}_nA_2, \ldots, {}_nA_r) \in \mathfrak{A}, \qquad {}_nA_j \in \mathfrak{A}_j.$$

Define $\psi\colon l^1(\mathbf{Z}^+, \mathfrak{A}, \alpha) \to \bigoplus_{j=1}^{r} l^1(\mathbf{Z}^+, \mathfrak{A}_j, \alpha_j)$ by

$$\psi\left(\sum_{n\geq 0} \delta_n \otimes {}_nA\right) = \left(\sum_{n\geq 0} \delta_n \otimes {}_nA_1, \ldots, \sum_{n\geq 0} \delta_n \otimes {}_nA_r\right).$$

ψ extends to an isometric isomorphism $\mathbf{Z}^+ \times_\alpha \mathfrak{A} \to \bigoplus_{j=1}^{r} \mathbf{Z}^+ \times_{\alpha_j} \mathfrak{A}_j$.

III.8. In what follows, the space of all maximal modular ideals in a Banach algebra, endowed with the hull-kernel topology, will be called the strong structure space [**9**]. We omit the proof of the following

LEMMA. *Let $\mathfrak{A}$ be a Banach algebra, with $\mathfrak{A} \cong \bigoplus_{i=1}^{r} \mathfrak{A}_i$, $r < \infty$, where $\mathfrak{A}_i$ is a Banach algebra $i = 1, 2, \ldots, r$. Then the strong structure space of $\mathfrak{A}$ is the disjoint union of the strong structure spaces of the $\mathfrak{A}_i$.*

III.9. Recall the Rudin (hull-kernel) topology on $\overline{\mathbf{D}} = \{z \in \mathbf{C}\colon |z| \leq 1\}$ determined by the algebra $A(\mathbf{D})$. The closed sets $V \subset \overline{\mathbf{D}}$ in this topology are of the following form:

(i) $V \cap \mathbf{D}$ is either finite or countable; if $V \cap \mathbf{D}$ is countable, say $\{\lambda_1, \lambda_2, \ldots\}$, then $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$.

(ii) $V \cap \partial\mathbf{D}$ is a closed subset (in the usual topology of $\partial\mathbf{D}$) of Lebesgue measure zero, and contains every accumulation point of $V \cap \mathbf{D}$ [**3**, p. 89].

Let K and N be positive integers, and let $\mathcal{M}$ and $\mathcal{N}$ be the strong structure spaces of $\mathcal{B}_K \otimes M_N(\mathbf{C})$ and $\mathcal{B}_K$, respectively. Then $\mathcal{M}$ and $\mathcal{N}$ are homeomorphic. The topology on $\mathcal{N}$ is given as follows: Set $S = \{s_0, s_1, \ldots, s_{K-1}\}$ and define an equivalence relation on $S \times \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ is endowed with the Rudin topology, by $(s, \lambda) \sim (s', \lambda')$ if

(i) $s = s'$ and $\lambda = \lambda' = 0$, or

(ii) $|\lambda| = |\lambda'| \neq 0$ and $(\lambda'/\lambda)^K = 1$.

Then $\mathcal{N} = S \times \overline{\mathbf{D}}/\sim$ with the quotient topology [**8**].

LEMMA. *For every positive integer K, the strong structure space of $\mathcal{B}_K$ is connected.*

PROOF. Suppose $\mathcal{N} = U \cup V$, with U, V open, and $U \cap V = \varnothing$. Let $U_0 = U \backslash (S \times \{0\})$, $V_0 = V \backslash (S \times \{0\})$; since $\{0\} \subset \overline{\mathbf{D}}$ is closed in the Rudin topology, $S \times \{0\}$ is closed in $S \times \overline{\mathbf{D}} \backslash \sim$, and so U_0 and V_0 are open. Furthermore, if U, V are nonempty, so are U_0, V_0. Let $p\colon S \times \overline{\mathbf{D}} \to S \times \overline{\mathbf{D}} \backslash \sim$ be the quotient map. Then $p^{-1}(U_0) = S \times U_1$ and $p^{-1}(V_0) = S \times V_1$ where U_1, V_1 are open sets in $\overline{\mathbf{D}} \backslash \{0\}$. Now

$$\varnothing = p^{-1}(U_0 \cap V_0) = p^{-1}(U_0) \cap p^{-1}(V_0) = (S \times U_1) \cap (S \times V_1) = S \times (U_1 \cap V_1).$$

Hence, $U_1 \cap V_1 = \varnothing$. Also,

$$\begin{aligned} S \times (\overline{\mathbf{D}} \backslash \{0\}) &= p^{-1}(U_0 \cup V_0) = p^{-1}(U_0) \cup p^{-1}(V_0) \\ &= (S \times U_1) \cup (S \times V_1) = S \times (U_1 \cup V_1). \end{aligned}$$

Therefore, $U_1 \cup V_1 = \overline{\mathbf{D}} \backslash \{0\}$ which implies that $\overline{\mathbf{D}} \backslash \{0\}$ is disconnected in the Rudin topology. But this topology is coarser than the Euclidean topology on $\overline{\mathbf{D}} \backslash \{0\}$. So the above contradicts the fact that $\overline{\mathbf{D}} \backslash \{0\}$ is connected in the Euclidean topology. Hence $\mathcal{N}$ is connected. □

III.10. COROLLARY. *For all positive integers K and N, the strong structure space of $\mathcal{B}_K \otimes M_N(\mathbf{C})$ is connected.*

III.11. LEMMA. *For all positive integers K and N, the only central idempotents of $\mathcal{B}_K \otimes M_N(\mathbf{C})$ are 0 and 1.*

PROOF. Let $E_\lambda\colon \mathcal{B}_K \otimes M_N(\mathbf{C}) \to M_{KN}(\mathbf{C}), \lambda \in \overline{\mathbf{D}}, \lambda \neq 0$, be the evaluations $E_\lambda(\hat{F}) = \hat{F}(\lambda)$, and $\hat{F} \in \mathcal{B}_K \otimes M_N(\mathbf{C})$ a central idempotent. Since E_λ is a homomorphism onto, it follows that $\hat{F}(\lambda)$ is in the center of $M_{KN}(\mathbf{C})$. Hence, $\hat{F}(\lambda) = w_\lambda I$, where I is the identity, and $w_\lambda \in \mathbf{C}$. Also, since $\hat{F}$ is idempotent, $w_\lambda = 0$ or $w_\lambda = 1$. This induces a continuous mapping $w\colon \overline{\mathbf{D}} \backslash \{0\} \to \{0, 1\}$. Hence, since $\overline{\mathbf{D}} \backslash \{0\}$ is connected, this implies $w_\lambda = 0$ or $w_\lambda = 1$ for every $\lambda \in \overline{\mathbf{D}}$, $\lambda \neq 0$. So that $\hat{F}(\lambda) = 0$ or $\hat{F}(\lambda) = I$ for every $\lambda \in \overline{\mathbf{D}}$, $\lambda \neq 0$. Since the elements of $\hat{F}$ are continuous in $\overline{\mathbf{D}}$, it follows that $\hat{F}(\lambda) = 0$ or $\hat{F}(\lambda) = I$ for $\lambda \in \overline{\mathbf{D}}$, and therefore $\hat{F} = 0$ or $\hat{F} = I$. □

III.12. THEOREM. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite-dimensional C^*-algebras, $\alpha \in \mathrm{Aut}(\mathfrak{A})$, $\beta \in \mathrm{Aut}(\mathfrak{B})$. Decompose $\mathfrak{A} \cong \bigoplus_{j=1}^{r} \mathfrak{A}_j, \mathfrak{B} \cong \bigoplus_{i=1}^{s} \mathfrak{B}_i$, where each $\mathfrak{A}_j \cong K_j M_{N_j}(\mathbf{C}), \mathfrak{B}_i \cong K_i' K_{N_i'}(\mathbf{C}), \alpha(\mathfrak{A}_j) = \mathfrak{A}_j, \beta(\mathfrak{B}_i) = \mathfrak{B}_i, 1 \leq j \leq r,\ 1 \leq i \leq s$, and α and β act transitively on the direct summands of $\mathfrak{A}_j$ and $\mathfrak{B}_i$, respectively. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathbf{Z}^+ \times_\beta \mathfrak{B}$ if and only if $r = s$, $N_j = N'_{\sigma(j)}$, $K_j = K'_{\sigma(j)}$ for some permutation σ of $\{1, 2, \ldots, r\}$.*

PROOF. If $r = s$, $N_j = N'_{\sigma(j)}$, $K_j = K'_{\sigma(j)}$ for some permutation σ of $\{1, 2, \ldots, r\}$, then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathbf{Z}^+ \times_\beta \mathfrak{B}$ by Proposition III.7.

Conversely, if $\mathbf{Z}^+ \times_\alpha \mathfrak{A} \cong \mathbf{Z}^+ \times_\beta \mathfrak{B}$ then by Theorem III.7

$$\bigoplus_{j=1}^{r} \mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \cong \bigoplus_{i=1}^{s} \mathcal{B}_{K_i'} \otimes M_{N_i'}(\mathbf{C}).$$

Lemma III.8 and Corollary III.10 imply $r = s$. So we have

$$\mathcal{D} = \bigoplus_{j=1}^{r} \mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \cong \bigoplus_{i=1}^{r} \mathcal{B}_{K_i'} \otimes M_{N_i'}(\mathbf{C}) = \mathcal{D}'.$$

Let ψ be an isomorphism between $\mathcal{D}$ and $\mathcal{D}'$. Consider $\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \subset \mathcal{D}$; this is an ideal in $\mathcal{D}$, and therefore its image under ψ is an ideal in $\mathcal{D}'$, and we may write $\psi(\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})) = \bigoplus_{i=1}^{r} J_i$, $J_i \subset \mathcal{B}_{K_i'} \otimes M_{N_i'}(\mathbf{C})$, J_i is an ideal, i.e., $\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \cong \bigoplus_{i=1}^{r} J_i$. This implies [**4**, p. 135] that there exist $e_i \in \mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})$, $1 \leq i \leq r$, orthogonal central idempotents such that $\sum_{i=1}^{r} e_i = 1$, and $e_i(\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})) \cong J_i$, $i = 1, 2, \ldots, r$. But by Lemma III.11 the only orthogonal central idempotents in $\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})$ are 0 and 1. Hence,

$$\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \cong J_{\sigma(j)} \subseteq \mathcal{B}_{K'_{\sigma(j)}} \otimes M_{N'_{\sigma(j)}}(\mathbf{C}).$$

Now the maximal modular ideals in $\mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C})$ have codimension N_j^2 or $K_j^2 N_j^2$. If $M \subset J_{\sigma(j)}$ is a maximal modular ideal, there exists $M' \otimes M_{N'_{\sigma(j)}}(\mathbf{C})$ a maximal modular ideal in $\mathcal{B}_{K'_{\sigma(j)}} \otimes M_{N'_{\sigma(j)}}(\mathbf{C})$ such that $M = J_{\sigma(j)} \cap (M' \otimes M_{N'_{\sigma(j)}}(\mathbf{C}))$, with $M' \subset \mathcal{B}_{K'_{\sigma(j)}}$ a maximal modular ideal. Therefore,

$$\begin{aligned} J_{\sigma(j)}/M &= J_{\sigma(j)}/[J_{\sigma(j)} \cap (M' \otimes M_{N'_{\sigma(j)}}(\mathbf{C}))] \\ &\cong [\mathcal{B}_{K'_{\sigma(j)}} \otimes M_{N'_{\sigma(j)}}(\mathbf{C})]/[M' \otimes M_{N'_{\sigma(j)}}(\mathbf{C})] \end{aligned}$$

by the Second Isomorphism Theorem. Hence, M has codimension $N'^2_{\sigma(j)}$ or $K'^2_{\sigma(j)} N'^2_{\sigma(j)}$. This implies $K_j = K'_{\sigma(j)}$ and $N_j = N'_{\sigma(j)}$. Repeating this process for every $j = 1, 2, \ldots, r$, we get the desired result. $\square$

III.13. COROLLARY. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite-dimensional C^*-algebras, $\alpha \in \mathrm{Aut}(\mathfrak{A}), \beta \in \mathrm{Aut}(\mathfrak{B})$. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is isomorphic to $\mathbf{Z}^+ \times_\beta \mathfrak{B}$ if and only if there is an isomorphism $\psi\colon \mathfrak{A} \to \mathfrak{B}$, and a unitary $U \in \mathfrak{B}$ such that $\beta \circ \psi = \mathrm{Ad}(U)(\psi \circ \alpha)$.*

PROOF. If $\mathbf{Z}^+ \times_\alpha \mathfrak{A} \cong \mathbf{Z}^+ \times_\beta \mathfrak{B}$ then by Theorem III.7

$$\bigoplus_{j=1}^{r} \mathcal{B}_{K_j} \otimes M_{N_j}(\mathbf{C}) \cong \mathbf{Z}^+ \times_\alpha \mathfrak{A} \cong \mathbf{Z}^+ \times_\beta \mathfrak{B} \cong \bigoplus_{i=1}^{s} \mathcal{B}_{K_i'} \otimes M_{N_i'}(\mathbf{C}),$$

so by Theorem III.12, $r = s$, $K_j = K'_{\sigma(j)}$, $N_j = N'_{\sigma(j)}$ for σ some permutation of $\{1, 2, \ldots, r\}$. Thus, there exist isomorphisms $u: \mathfrak{A} \to \bigoplus_{j=1}^{r} \mathcal{D}_j$ and $v: \mathfrak{B} \to \bigoplus_{j=1}^{r} \mathcal{D}_j$ with $\mathcal{D}_j \cong K_j M_{N_j}(\mathbf{C})$, and α and β act transitively on the direct summands of $\mathcal{D}_j$; that is, there exist unitaries $V_{ij} \in M_{N_j}(\mathbf{C})$, $j = 1, 2, \ldots, r$; $i = 0, 1, \ldots, K_j - 1$, such that $[v \circ \beta](v^{-1} \circ u) = \mathrm{Ad}(V)(u \circ \alpha)$ where $V = V_{01} \oplus V_{11} \oplus \cdots \oplus V_{K_1-1,1} \oplus V_{02} \oplus V_{12} \oplus \cdots \oplus V_{K_2-1,2} \oplus V_{0r} \oplus V_{1r} \oplus \cdots \oplus V_{K_r-1,r}$. Let $U = v^{-1}(V) \in \mathfrak{B}$, and $\psi = v^{-1} \circ u$; then U is unitary and ψ is an isomorphism. It follows that $\beta \circ \psi = \mathrm{Ad}(U)(\psi \circ \alpha)$.

The converse follows easily from III.4.

Whenever $\mathfrak{A}$ and $\mathfrak{B}$ are commutative, $K_j = 1$ in Theorem III.7. In this case $\mathbf{Z}^+ \times_\alpha \mathfrak{A} \cong \mathbf{Z}^+ \times_\beta \mathfrak{B}$ if and only if there exists an isomorphism $\psi: \mathfrak{A} \to \mathfrak{B}$, such that $\beta \circ \psi = \psi \circ \alpha$.

III.14. COROLLARY. *Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra, $\alpha \in \mathrm{Aut}(\mathfrak{A})$. Then $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$ is strongly semisimple.*

III.151. EXAMPLE. Isomorphism of C^*-crossed products $\mathbf{Z}^+ \times_\alpha \mathfrak{A}$, and $\mathbf{Z} \times_\beta \mathfrak{B}$ does not imply that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Take $\mathfrak{A} = \mathbf{C}^2$, and $\alpha \in \mathrm{Aut}(\mathfrak{A})$, $\alpha(z_1, z_2) = (z_2, z_1)$. $\mathfrak{B} = M_2(\mathbf{C})$, β the identity automorphism; then both $\mathbf{Z} \times_\alpha \mathfrak{A}$ and $\mathbf{Z} \times_\beta \mathfrak{B}$ are isomorphic to $C(\mathbf{T}) \otimes M_2(\mathbf{C})$.

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