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REMARK ON THE GENERALIZED PUTNAM–FUGLEDE THEOREM

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ABSTRACT. If A^* is M-hyponormal, B is dominant, and if $CA = BC$, then $CA^* = B^*C$.

Let $\mathscr{H}$ be a Hilbert space. An operator $A \in \mathscr{B}(\mathscr{H})$ is dominant if range$(A - \lambda I) \subset$ range$(A - \lambda I)^*$ for all $\lambda \in \sigma(A)$, or, equivalently, for all $\lambda \in \mathbf{C}$ because

$$\operatorname{range}(A - \lambda I) = \operatorname{range}(A - \lambda I)^* = \mathscr{H}$$

for $\lambda \notin \sigma(A)$. By the result of Douglas [**1**], it is equivalent to the existence of a real number M_λ for each $\lambda \in \mathbf{C}$ such that

$$\|(A - \lambda I)^* x\| \leqslant M_\lambda \|(A - \lambda I)x\|$$

for all $x \in \mathscr{H}$. If there is a constant M such that $M_\lambda \leqslant M$ for all $\lambda \in \mathbf{C}$, A is called M-hyponormal, and if $M = 1$, A is hyponormal.

Stampfli and Wadhwa [**5**] showed that if $A^* \in \mathscr{B}(\mathscr{H})$ is hyponormal, $B \in \mathscr{B}(\mathscr{K})$ is dominant, $C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ is injective and has dense range, and if $CA = BC$, then A and B are normal.

Radjabalipour [**3**] improved this result by allowing A^* to be M-hyponormal. Of course, the condition that A and B are normal allows one to conclude immediately by the usual Putnam–Fuglede theorem that $CA^* = B^*C$.

Moore, Rogers and Trent [**2**] showed that, if A^* and B are both M-hyponormal, the conclusion $CA^* = B^*C$ can be reached with no restrictions on C at all.

We show that this result can also be improved by allowing B to be dominant.

We need the following two lemmas:

LEMMA 1 [**4**]. *Let A be dominant and let $\mathscr{M}$ be an invariant subspace of A for which $A|\mathscr{M}$ is normal. Then $\mathscr{M}$ reduces A.*

LEMMA 2. *The restriction $A|\mathscr{M}$ of the dominant (respectively, M-hyponormal) operator A on $\mathscr{H}$ to an invariant subspace $\mathscr{M}$ of A is dominant (respectively, M-hyponormal).*

PROOF. Let $P_{\mathscr{M}}$ be the orthogonal projection from $\mathscr{H}$ to $\mathscr{M}$. Then

$$\begin{aligned}\|(A|\mathscr{M} - \lambda I)^* x\| &= \|P_{\mathscr{M}}(A - \lambda I)^* x\| \leqslant \|(A - \lambda I)^* x\| \\ &\leqslant M_\lambda \,(\text{resp. } M)\|(A - \lambda I)x\| = M_\lambda \,(\text{resp. } M)\|(A|\mathscr{M} - \lambda I)x\|\end{aligned}$$

for each $\lambda \in \mathbf{C}$ and all $x \in \mathscr{M}$.

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THEOREM. *If $A^* \in \mathscr{B}(\mathscr{H})$ is M-hyponormal, $B \in \mathscr{B}(\mathscr{K})$ is dominant, and if $CA = BC$ for $C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, then $CA^* = B^*C$.*

PROOF. From $CA = BC$ we know that $(\ker C)^{\perp}$ and $(\operatorname{range} C)^{-}$ are invariant subspaces of A^* and B, respectively. Hence $A^*|(\ker C)^{\perp}$ is M-hyponormal and $B|(\operatorname{range} C)^{-}$ is also dominant by Lemma 2. By the decompositions $\mathscr{H} = (\ker C)^{\perp} \oplus (\ker C)$ and $\mathscr{K} = (\operatorname{range} C)^{-} \oplus \{(\operatorname{range} C)^{-}\}^{\perp}$, we have

$$A = \begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $A_1^* = A^*|(\ker C)^{\perp}$ is M-hyponormal, $B_1 = B|(\operatorname{range} C)^{-}$ is dominant and C_1 is injective with dense range. We obtain $C_1A_1 = B_1C_1$ from $CA = BC$. Hence, A_1 and B_1 are normal by [**3**, Theorem 3(a)] and $C_1A_1^* = B_1^*C_1$ by the usual Putnam–Fuglede theorem. Then, by Lemma 1, $(\ker C)^{\perp}$ and $(\operatorname{range} C)^{-}$ reduces A^* and B to normal operators, respectively. Therefore, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

Hence we obtain $CA^* = B^*C$.

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