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A LOCAL KERNEL PROPERTY OF CLOSED DERIVATIONS ON $C(I \times I)$

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ABSTRACT. In this note we show a local behavior of closed derivations on $C([0,1] \times [0,1])$, which is essentially different from one-dimensional derivations. Roughly speaking, any closed derivations on $C([0,1] \times [0,1])$ has a nonconstant kernel locally.

Our concerns are unbounded densely defined derivations on the Banach algebra $C(\Omega)$ of continuous real-valued functions on a compact Hausdorff space Ω.

A (closed) *derivation* δ on $C(\Omega)$ means a (closed) linear map in $C(\Omega)$ defined on a dense subalgebra $\mathscr{D}(\delta)$ which satisfies the derivation property, i.e., $\delta(fg) = \delta(f)g + f\delta(g)$ for every f, g in $\mathscr{D}(\delta)$. Without loss of generality, we shall assume that the unit function $\mathbf{1}$ belongs to $\mathscr{D}(\delta)$ (so that $\delta(\mathbf{1}) = 0$ follows from the derivation property). The kernel and range of δ are denoted by ran δ and ker δ, respectively.

A closed subset E in Ω is called a *self-determining set* for δ if

$$\delta(f)|_E = 0 \quad \text{whenever } f \in \mathscr{D}(\delta) \text{ and } f|_E = 0,$$

where $f|_E$ means the restriction of f to E. If E is a self-determining set, the formula $\delta_E(f|_E) = \delta(f)|_E$ defines a derivation δ_E with domain $\{f|_E;\, f \in \mathscr{D}(\delta)\}$. It is known (see [**1**, Lemma 4.1 or **2**, Lemma 1.1.10]) that if δ is a closed derivation, for every open subset U in Ω the closure $\overline{U}$ is a self-determining set for δ and, moreover, $\delta_{\overline{U}}$ is closable. For closable derivation $\delta_{\overline{U}}$, the closure will be written by $\tilde{\delta}_{\overline{U}}$.

Closed derivations on the unit interval $I = [0,1]$ have been studied by various authors and recently the structure has been made clear (see references). On the contrary, even the derivations on $C(I \times I)$ are almost left untouched.

The following theorem gives a property of derivations on $C(I \times I)$ which clearly holds for partial derivatives, but which does not hold for derivations on $C(I)$.

THEOREM. *For any closed derivation δ on $C(I \times I)$ with* ran $\delta = C(I \times I)$ *and any open subset V in $I \times I$, there is a nonempty connected open subset U contained in V such that* ker $\delta_{\overline{U}}$ *contains some nonconstant functions in $\mathscr{D}(\delta_{\overline{U}})$.*

PROOF. Without loss of generality we may assume that V is connected. We consider two cases.

Case 1. There exists at least one nonempty open connected subset W of V such that $\delta_{\overline{W}}$ is not closed (but closable). In this case, it is easily shown that ker $\tilde{\delta}_{\overline{W}}$

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contains a nonconstant function F. In fact, there is a function G_1 in $\mathscr{D}(\tilde{\delta}_{\overline{W}})$, but not in $\mathscr{D}(\delta_{\overline{W}})$. To such G_1 a function G_2 in $\mathscr{D}(\delta_{\overline{W}})$ can be selected so as to satisfy $\delta_{\overline{W}}G_2 = \tilde{\delta}_{\overline{W}}G_1$, because the assumption that $\operatorname{ran}\delta = C(I \times I)$ and the normality of the compact Hausdorff space $I \times I$ imply $\operatorname{ran}\delta_{\overline{W}} = C(\overline{W})$. Then the function $F = G_1 - G_2$ is a desired function.

Therefore, there is a sequence of functions $\{f_n\}$ in $\mathscr{D}(\delta)$ such that

$$f_n|_{\overline{W}} \to F \quad \text{and} \quad \delta f_n|_{\overline{W}} \to 0 \qquad (n \to \infty).$$

Let U be any open connected subset in W such that $\overline{U} \subset W$ and $F|_{\overline{U}}$ is not constant. It is known (see [**1**, Lemma 2.3]) that there exists a function h in $\mathscr{D}(\delta)$ such that $h = 1$ on $\overline{U}$ and $\operatorname{supp} h \subset \overline{W}$. Then it is easily checked that

$$hf_n \to f \quad \text{and} \quad \delta(hf_n) \to g \qquad (n \to \infty),$$

where

$$f = \begin{cases} hF & \text{on } \overline{W}, \\ 0 & \text{on } \overline{W}^c, \end{cases} \quad \text{and} \quad g = \begin{cases} 0 & \text{on } \overline{U}, \\ (\delta h)F & \text{on } \overline{W} \cap \overline{U}^c, \\ 0 & \text{on } \overline{W}^c. \end{cases}$$

By closedness of δ, f belongs to $\mathscr{D}(\delta)$ and $\delta f = g$, so that $f|_{\overline{U}}$ belongs to $\mathscr{D}(\delta_{\overline{U}})$ and is not constant on $\overline{U}$, moreover, $\delta_{\overline{U}}(f|_{\overline{U}}) = \delta f|_{\overline{U}} = g|_{\overline{U}} = 0$.

Case 2. The derivation $\delta_{\overline{U}}$ is always a closed derivation for every open connected subset U of V. In particular, $\delta_{\overline{V}}$ is a closed derivation with $\operatorname{ran}\delta_{\overline{V}} = C(\overline{V})$. Hereafter $\delta_{\overline{V}}$ will be written by δ for the simplicity of notation.

Now suppose that our assertion is not true. Then for any open connected subset U of V, $\ker\delta_{\overline{U}} = \{\lambda\mathbf{1};\ \lambda \text{ real}\}$ holds. In particular, $\ker\delta = \{\lambda\mathbf{1};\ \lambda \text{ real}\}$. Take any point x_0 in $\overline{V}$, and let $\mathscr{D}_0 = \{f \in \mathscr{D}(\delta);\ f(x_0) = 0\}$. Then clearly δ is a closed linear one-to-one map from $\mathscr{D}_0$ onto $C(\overline{V})$, and δ^{-1} is a bounded linear operator on $C(\overline{V})$ by the closed graph theorem. Hence, for any fixed x in $\overline{V}$, the map $C(V) \ni F \mapsto (\delta^{-1}F)(x)$ is a bounded linear functional defined on $C(\overline{V})$, and by the Riesz representation theorem there exists a unique signed measure μ_x on $\overline{V}$ such that

$$(1) \qquad (\delta^{-1}F)(x) = \int_{\overline{V}} F\,d\mu_x \quad \text{for every } F \in C(\overline{V}).$$

Now take any connected open subset U whose closure contains the points x and x_0. Since the closed derivation $\delta_{\overline{U}}$ also satisfies that $\operatorname{ran}\delta_{\overline{U}} = C(\overline{U})$ and $\ker\delta_{\overline{U}} = \{\lambda\mathbf{1};\ \lambda \text{ real}\}$, we can repeat the preceding argument with $\delta_{\overline{U}}$ instead of δ, so that for any point y in $\overline{U}$ there exists a unique signed measure $\mu_{y,\overline{U}}$ on $\overline{U}$ such that

$$(2) \qquad (\delta_{\overline{U}})^{-1}F(y) = \int_{\overline{U}} F\,d\mu_{y,\overline{U}} \quad \text{for every } F \in C(\overline{U}).$$

This measure $\mu_{y,\overline{U}}$ can be considered as a measure on $\overline{V}$ by defining it to be zero outside $\overline{U}$. Then (2) can be written as

$$(2)' \qquad (\delta_{\overline{U}})^{-1}F|_{\overline{U}}(y) = \int_{\overline{V}} F\,d\mu_{y,\overline{U}} \quad \text{for every } F \in C(\overline{V}) \text{ and } y \in \overline{U}.$$

Here, if we put $f = \delta^{-1}F$, then $\delta f = F$ and $\delta_{\bar U}(f|_{\bar U}) = F|_{\bar U}$, and hence

$$(\delta_{\bar U})^{-1}F|_{\bar U}(y) = f(y) = (\delta^{-1}F)(y)$$

for any $y \in \bar U$. Accordingly, (1) and (2)′ with $y = x$ imply

$$\int_{\bar V} F\,d\mu_x = \int_{\bar V} F\,d\mu_{x,\bar U}$$

for every $F \in C(\bar V)$, from which follows $\mu_x = \mu_{x,\bar U}$. In particular, $\operatorname{supp}\mu_x$ is contained in $\bar U$. Since U can be selected arbitrarily so as to contain the points x and x_0, μ_x must be an atomic measure with $\operatorname{supp}\mu_x \subset \{x_0, x\}$ (cf. Remark following the proof). Therefore, (1) transforms into the form

$$f(x) = \delta f(x_0)\mu_x(x_0) + \delta f(x)\mu_x(x) \quad \text{for every } f \in \mathscr{D}_0.$$

Since the value $\delta f(x_0)$ can be chosen independently for the values $f(x)$ and $\delta f(x)$, the point-mass $\mu_x(x_0)$ must be zero. Moreover, it can be easily shown that the point-mass $\mu_x(x)$ is also zero, by the derivation property. It follows that $\mathscr{D}_0$ is trivial, which leads to a contradiction. This concludes the proof.

REMARK. (1) The proof of Theorem remains valid in the case that the space $I \times I$ is replaced by a compact Hausdorff space Ω satisfying the following property:

(∗) For any three points x, y and z in Ω, there is an open connected subset U such that $\{x, y\} \subset U$ and $z \notin \bar U$.

Clearly the space I does not possess the property.

(2) While the one-dimensional torus T also satisfies (∗), it is known (e.g. [**6**, Theorem 3.2.6]) that for any closed derivation on $C(T)$ the conclusion in the Theorem is invalid. Hence, it follows that the range of any closed derivation on $C(T)$ never coincides with $C(T)$.

We conclude with an example which is simple and relevant to our subjects.

EXAMPLE. The derivation on $C(I \times I)$ is defined as

$$f(r,\theta) = \partial f(r,\theta)/\partial r$$

for f in $\mathscr{D}(\delta)$, where (r, θ) stand for the polar coordinates in $I \times I$. Every function f in $\mathscr{D}(\delta)$ has every directional derivative at the origin, that is

$$\lim_{\Delta r \to 0} \frac{f(\Delta r, \theta) - f(0)}{\Delta r}$$

exists for every $\theta \in [0, \pi/2]$, and all the values must coincide.

Then δ is certainly closed and $\operatorname{ran}\delta = C(I \times I)$. Moreover, it is easily seen that $\ker\delta = \{\lambda\mathbf{1};\ \lambda \text{ real}\}$ and $\ker\delta_{\bar U}$ contains many nonconstant functions for every open subset U whose closure does not contain the origin.

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