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CHARACTERIZATIONS OF BAIRE* 1 FUNCTIONS IN GENERAL SETTINGS

DARWIN E. PEEK

ABSTRACT. Baire* 1 functions from $[0,1]$ to R were defined by R. J. O'Malley. For a general topological space X, a function $f\colon X \to R$ will be said to be Baire* 1 if and only if for every nonempty closed subset H of X, there is an open set U such that $U \cap H \neq \varnothing$ and $f|H$ is continuous on U. Several characterizations of Baire* 1 functions are found by altering the well-known Baire 1 characterization: If H is a nonempty *closed subset* of the domain of f, then $f|H$ has a point where $f|H$ is continuous. These conditions simply replace "*closed subset* of the preceding characterization with "*subset*", "*countable subset*" or "*dense-in-itself subset*". The relationships of these characterizations are examined with the domain of f being various spaces. The independence of these conditions from the discrete convergence condition described by Á. Császár and M. Laczkovich is discussed.

1. Introduction. All functions $f\colon X \to R$ will be from a topological space X into the set R of real numbers. The mutual relationships of the following conditions (a)–(e) will be examined. Condition (a) is a generalization of the Baire* 1 condition as defined by R. J. O'Malley [7, p. 211]. Condition (e) was defined by Á. Császár and M. Laczkovich [**2**, p. 463].

(a) The function f is Baire* 1, that is, for every nonempty closed subset H of X, there is an open set U such that $U \cap H$ is not empty and $f|H$ is continuous on U.

(b) If H is a nonempty subset of X, then $f|H$ has a point where $f|H$ is continuous.

(c) If H is a nonempty dense-in-itself subset of X, then $f|H$ has a point where $f|H$ is continuous.

(d) If H is a nonempty countable subset of X, then $f|H$ has a point where $f|H$ is continuous.

(e) The function f is the discrete limit of a sequence of continuous functions as defined by Császár and Laczkovich.

§2 will show that (a)–(c) are equivalent for any domain. The Baire* 1 condition (a) is shown to imply the "countable" condition (d) in any domain, while the converse is demonstrated to be false. §3 does give a converse theorem when the domain is hereditarily separable. §4 discusses the independence of Császár and Laczkovich's condition (e) from the other conditions.

2. General equivalences to Baire* 1. Theorem 1 and Comment 2 will state the only implications, one to another, of the preceding conditions where the domain of f is an

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arbitrary space. Example 3 will show that the converse of Comment 2 is false. The failure of the other implications will be discussed in §§ 4 and 5.

THEOREM 1. *Suppose f is a function from a topological space X into the set of real numbers R. The following statements are equivalent*:

(a) *The function f is Baire* 1.*

(b) *If H is a nonempty subset of X, then $f|H$ has a point where $f|H$ is continuous.*

(c) *If H is a nonempty dense-in-itself subset of X, then $f|H$ has a point where $f|H$ is continuous.*

PROOF. *Proof that* (a) *implies* (b). Suppose (a). Suppose (b) is false. There is a nonempty subset H of X such that $f|H$ is discontinuous at each of its points. Since f is Baire* 1, there is an open set U such that $U \cap \mathrm{Cl}(H) \neq \varnothing$ and $f|\mathrm{Cl}(H)$ is continuous on U. U contains a point x of H. Therefore, $f|\mathrm{Cl}(H)$ is continuous at x. Therefore, $f|H$ is continuous at x. Contradiction. Therefore, (a) implies (b).

Proof that (b) *implies* (c). Trivial.

Proof that (c) *implies* (a). Suppose (c). Suppose (a) is false. Therefore, there is a closed nonempty subset H of X such that if J is an open set such that $H \cap J \neq \varnothing$, then $f|H \cap J$ has a point where $f|H$ is not continuous. Let D denote the subset of H where $f|H$ is not continuous. If D is not dense-in-itself, then D contains an isolated point x and $f|D$ is continuous at x. D is nonempty. Therefore, if D is dense-in-itself, then $f|D$ has a point x where $f|D$ is continuous. In either case $x \in D$. Therefore, $f|H$ is not continuous at x. Let V denote an open set containing $f(x)$ such that, if U is an open set containing x, then there is a $y \in U \cap H$ such that $f(y) \notin \mathrm{Cl}(V)$. There is an open set W containing x such that if $y \in W \cap D$ then $f(y) \in V$. There is a $z \in W \cap H$ such that $f(z) \notin \mathrm{Cl}(V)$. Therefore, $z \notin D$. Therefore, $f|H$ is continuous at z. Therefore, there is an open set S containing z such that if $y \in H \cap (S \cap W)$ then $f(y) \in R - \mathrm{Cl}(V)$. Therefore, $y \notin D$. Therefore, $S \cap W$ is an open set such that $H \cap (S \cap W)$ is not empty and $f|H \cap (S \cap W)$ is continuous. Contradiction. Therefore, (c) implies (a).

Comment 2. It is now obvious that statement (a) implies statement (d).

The following example shows that the countable condition (d) does not, in general, imply the Baire* 1 condition (a).

EXAMPLE 3. Let τ denote the countable complement topology on the set R of real numbers. That is, the nonempty open sets of τ are the complements of countable sets. Let A and B denote two totally imperfect (i.e. contain no nonempty perfect subsets) nonempty subsets of R such that $A \cap B = \varnothing$ and $A \cup B = R$ [**5**, pp. 201–202]. Let f denote the characteristic function of A on R. Suppose H is a nonempty countable subset of R and $x \in H$. Let U denote $(R - H) \cup \{x\}$. U is open in τ and $H \cap U = \{x\}$. Therefore, $f|H$ is continuous at x. Therefore, f satisfies condition (d). Let K denote a perfect (in the Euclidean topology) subset of R. $f|K$ is totally discontinuous on K. Therefore, f does not satisfy condition (a). Therefore, (d) does not imply (a) when the domain of f is the countable complement topology.

3. Hereditarily separable domains. Example 3 demonstrated that the converse of Comment 2 is not true. When the domain of the function is hereditarily separable the converse of Comment 2 does, in fact, hold and this will be shown in Theorem 5.

Comment 4. If X is hereditarily separable and Y is a second countable, then $X \times Y$ is hereditarily separable.

THEOREM 5. *Suppose X is a hereditarily separable space and f is a function from X into the set of real numbers R. The following two statements are equivalent*:

(a) *The function f is* Baire* 1.

(d) *If H is a nonempty countable subset of X, then $f|H$ has a point where $f|H$ is continuous.*

PROOF. *Proof that* (a) *implies* (d). By Theorem 1, (a) implies (b), and (b) implies (d) here.

Proof that (d) *implies* (a). Suppose (d). Suppose (a) is false. There is a closed nonempty subset H of X such that if U is an open set such that $H \cap U \neq \varnothing$, then $f|H \cap U$ has a point where $f|H \cap U$ is not continuous. Let D denote the subset of H where $f|H$ is not continuous. The function $f|D$ is a subset of $X \times R$. X is hereditarily separable and R is second countable. Therefore, by Comment 4, $f|D$ is separable. Since $f|D$ is also a function, there is a countable nonempty subset C of D such that $f|C$ is dense in $f|D$ and C is dense in D. Since C is countable, $f|C$ has a point where $f|C$ is continuous. Since $f|C$ is dense in $f|D$, $f|D$ must also be continuous at $(x, f(x))$. The function $f|H$ is not continuous at $(x, f(x))$. Let V denote a neighborhood of $f(x)$ such that if U is an open subset of X containing x, then there is a $y \in H \cap U$ such that $f(y) \notin \mathrm{Cl}(V)$. Let J denote an open subset of X containing x such that if $y \in D \cap J$, then $f(y) \in V$. Let z denote an element of $H \cap J$ such that $f(z) \notin \mathrm{Cl}(V)$. $R - \mathrm{Cl}(V)$ is an open set containing $f(z)$. Since z is not in D, $f|H$ is continuous at z. There is an open subset W of X containing z such that, if $y \in W \cap H$, then $f(y) \in R - \mathrm{Cl}(V)$. Therefore, if $y \in (W \cap J) \cap H$ then $f(y) \in R - \mathrm{Cl}(V)$ and, consequently, $y \notin D$. Therefore, $f|H$ is continuous at y. Therefore, $f|H$ is continuous on the open set $W \cap J$. Since $z \in W \cap H$ and $z \in J \cap H$, $(W \cap J) \cap H$ is nonempty. Therefore, $W \cap J$ is an open set such that $(W \cap J) \cap H \neq \varnothing$ and $f|(W \cap J) \cap H$ is continuous. Contradiction. Therefore, f is a Baire* 1 function. Therefore, (d) implies (a).

4. Discrete convergence of sequences of continuous functions. The function f: $X \to R$ is the discrete limit of a sequence of functions $\{f_n\}$ means for every $x \in X$ there exists n_0 such that, if $n \geqslant n_0$, $f(x) = f_n(x)$. The functions f: $X \to R$ are the equal limit of a sequence of functions $\{f_n\}$ means there is a sequence of positive numbers $\{\varepsilon_n\} \to 0$ such that for every $x \in X$ there exists n_0 such that, if $n \geqslant n_0$, then $|f(x) - f_n(x)| < \varepsilon_n$ [**2**, p. 463]. Császár and Laczkovich [**2**, p. 471] give an example in which there are Baire* 1 functions that are not discrete limits of sequences of continuous functions. In other words, in general, (a) does not imply (e). The following example shows that, in general, the converse is also not true, as well as that (e) does not imply (d).

EXAMPLE 6. For each rational number p, let $f(p) = 1/n$ where $p = m/n$ and $(m, n) = 1$. The function $f|Q$ is totally discontinuous on Q. Therefore, f does not satisfy (d). Since Comment 2 states that (a) implies (d), f cannot satisfy (a). The function f is the limit of a sequence $\{f_n\}$ of continuous functions. Q is the union of a countable sequence $\{A_i\}$ of singleton sets. For each positive integer i, $\{f_n|A_i\}$ converges to $f|A_i$ uniformly. Consequently, by Theorem 5.1 of [**3**, p. 66], $\{f_n\}$ converges equally to f. By Theorem 1 of [**2**, p. 463], f is the discrete limit of a sequence of continuous functions. Therefore, f satisfies condition (e).

Example 3 shows that (d) does not imply (e) in general. For suppose the function f defined in Example 3 is the discrete limit of a sequence $\{f_n\}$ of continuous (in the countable complement topology) functions. There is a positive integer n such that $B = \{x: f_n(x) = 1\}$ and $C = \{x: f_n(x) = 0\}$ are both uncountable. Every nonempty open (in the countable complement topology) set contains points of B and points of C. Consequently, f_n is totally discontinuous in the countable complement topology. Contradiction. Therefore, in general, (d) does not imply (e).

Császár and Laczkovich, in Theorem 15 of [**2**, p. 470], show that (a) implies (e) for a σ-compact, perfectly normal Hausdorff space. Gerlits shows in Theorem 2 of [**4**, p. 147] the same implication for a subparacompact and perfectly normal Tychonoff space.

That (e) implies (a) in a complete metric space follows from Theorem 3 of [**4**, p. 149].

5. Conclusion. Conditions (a)–(c) were shown to be equivalent. Condition (a) was shown to imply (d). The function in Example 3 satisfies condition (d), but not (a) and not (e). There are functions in Császár and Laczkovich's example that satisfy conditions (a) and (d), but not (e). Example 6 defines a function that does not satisfy (d) and (a), but does satisfy (e). There are three remaining cases involving (a), (d), and (e). The first case is whether there is a function satisfying (a), (d), and (e). This case can be settled by letting f be any continuous function. The second case is whether there is a function that satisfies none of the three conditions (a), (d), or (e). The characteristic function of Example 3 using the Euclidean topology settles this question. The last case is unanswered. This case asks whether, in general, "do conditions (d) and (e) together imply (a)?"

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DEPARTMENT OF MATHEMATICS, TRINITY UNIVERSITY, SAN ANTONIO, TEXAS 78284