PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 4, December 1985

PROPERTY (H) IN LEBESGUE-BOCHNER FUNCTION SPACES

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ABSTRACT. We prove that if a Banach space X has the property (HR) and if l_1 is not isomorphic to a subspace of X, then every point on the unit sphere of X is a denting point of the closed unit ball. We also prove that if X has the above property, then $L^p(\mu, X)$, $1 < p < \infty$, has the property (H).

1. Introduction. A Banach space is said to have the *property* (H) **[1]** (known also as the *Radon-Riesz property* **[10]** or the *Kadec-Klee property* **[3]**) if every sequence of norm-one elements that converges weakly to a norm-one element converges in norm. M. A. Smith and B. Turett **[10]** proved that if (Ω, Σ, μ) is a measure space that is not purely atomic and $L^p(\mu, X)$, $1 < p < \infty$, has the property (H), then X is strictly convex (R). They posed the following problem:

Question. If X is a strictly convex Banach space with property (H), does $L^p(\mu, X)$, $1 < p < \infty$, have property (H)?

The purpose of this paper is to show that the answer to the above question is affirmative when l_1 is not isomorphic to a subspace of X. Let us recall some other definitions.

(G) Every element in the unit sphere is a denting point of the closed unit ball, i.e., if $\|x_0\| = 1$ then $x_0 \notin \overline{\text{co}}(M(x_0, \varepsilon))$ for all $\varepsilon > 0$ where $M(x_0, \varepsilon) = \{x \in X\colon \|x\| \leqslant 1 \text{ and } \|x - x_0\| \geqslant \varepsilon\}$.

(MLUR) If, for each $\varepsilon > 0$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(z, \varepsilon) > 0$ such that if x and y are in X with $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$ then $\|x + y - 2z\| \geqslant \delta(z, \varepsilon)$.

The properties (G) and (HR) were introduced by Ky Fan and I. Glicksberg **[2]**. They proved that (G) implies (HR) and that (G) and (HR) are equivalent when X is reflexive. Midpoint local uniform convexity (MLUR) was introduced by K. W. Anderson (see **[10]**). It is easy to see that it is equivalent to the following property:

If, for each $\varepsilon > 0$, $0 < \alpha \leqslant \beta < 1$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(z, \varepsilon, \alpha, \beta) > 0$ such that if x and y are in X with $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and $\|y - x\| > \varepsilon$ then $\|\nu x + (1 - \nu)y - z\| > \delta(z, \varepsilon, \alpha, \beta)$ when $\alpha \leqslant \nu \leqslant \beta$.

Received by the editors December 11, 1984 and, in revised form, February 8, 1985.
1980 *Mathematics Subject Classification*. Primary 46B20, 46E40; Secondary 46B22.
Key words and phrases. Property (H), Lebesgue-Bochner function spaces.
[1]Research of this author was partially supported by NSF grant DMS-8201635.

0002-9939/85 $1.00 + $.25 per page

It is known that if X has the property (G), then X is (MLUR). On the other hand, M. A. Smith [**9**] renormed l_1 such that it has the property (HR), but it is not (MLUR). M. I. Kadec [**4**] used H. P. Rosenthal's l_1 theorem to show that if X has the property (HR) and if X does not contain l_1, then X is (MLUR). We prove the following two theorems.

THEOREM 1. *If X has the property* (HR), *and if l_1 is not isomorphic to a subspace of X, then X has the property* (G).

THEOREM 2. *Suppose X has the property* (G). *Then $L^p(\mu, X)$, $1 < p < \infty$, has the property* (H).

For more geometrical properties between X and $L^p(\mu, X)$, we suggest the reader consult [**6** and **10**]. We wish to thank Professor B. Turett for informing us of the paper of Kadec [**4**].

2. Proof of Theorem 1. We may assume that X is separable. Suppose X does not have the property (G). Then there are x_0 with $\|x_0\| = 1$ and $\varepsilon > 0$ such that $x_0 \in \overline{\text{co}} M(x_0, \varepsilon)$ where $M(x_0, \varepsilon) = \{x \in X\colon \|x - x_0\| \geqslant \varepsilon \text{ and } \|x\| \leqslant 1\}$. If x_0 belongs to the weak closure of $M(x_0, \varepsilon)$, since l_1 is not isomorphic to a subspace of X, then there is a sequence (x_n) in $M(x_0, \varepsilon)$ which converges to x_0 weakly [**5**]. Since X has the property (H), (x_n) converges to x_0 in norm. This contradicts the fact $\|x_n - x_0\| \geqslant \varepsilon$ for all n. So there exist $\delta > 0$ and $f_1, f_2, \ldots, f_n$ in X^* such that, if $x \in M(x_0, \varepsilon)$, then $f_k(x) < f_k(x_0) - \delta$ for some $k \leqslant n$. Let

$$M_k(x_0, \varepsilon) = \{x\colon x \in M(x_0, \varepsilon) \text{ and } f_k(x) < f_k(x_0) - \delta\}.$$

If $x_0 \notin \overline{\text{co}}(M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon))$, then there exist $\delta' > 0$ and $f \in X^*$ such that if $x \in M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon)$ then $f(x) < f(x_0) - \delta'$. Hence, in this case, $M(x_0, \varepsilon) = \bigcup_{i=3}^n M_i(x_0, \varepsilon) \cup M_f(x_0, \varepsilon)$ where $M_f(x_0, \varepsilon) = \{x\colon x \in M(x_0, \varepsilon), f(x) < f(x_0) - \delta'\}$. So we may assume that

$$x_0 \in \overline{\text{co}}\, M(x_0, \varepsilon) = \overline{\text{co}}\,(M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon))$$

and there exist sequences (x_n) in $M_1(x_0, \varepsilon)$, (y_n) in $M_2(x_0, \varepsilon)$ and (α_n) with $0 \leqslant \alpha_n \leqslant 1$ such that $\lim_{n\to\infty} \alpha_n x_n + (1 - \alpha_n) y_n = x_0$. Obviously, $0 < \lim \alpha_n < 1$. This contradicts the fact that X is (MLUR). So X has the property (G). Q.E.D.

3. Proof of Theorem 2. Let (f_n) be a norm-one sequence in $L^p(\mu, X)$ which converges weakly to a norm-one element f. Let g be an element in $L^p(\mu, X)^*$ such that $g(f) = 1 = \|g\|$. Then

$$1 \geqslant \tfrac{1}{2}\big\|\|f_n(\cdot)\| + \|f(\cdot)\|\big\|_{L^p(\mu)} \geqslant \tfrac{1}{2}\|f_n + f\|_{L^p(\mu, X)} \geqslant \tfrac{1}{2} g(f_n + f).$$

Since this last term converges to 1 and $L^p(\mu)$, $1 < p < \infty$, is uniformly convex, $\|f_n(\cdot)\|$ converges to $\|f(\cdot)\|$ in $L^p(\mu)$. By passing to subsequence and perturbing the sequence (f_n), we may assume that $\|f_n(t)\| = \|f(t)\|$ for all $n \in \mathbf{N}$ and $t \in \Omega$. Let

$$d(n, k) = \{t\colon \|f_n(t) - f(t)\| > \|f(t)\|/k\}.$$

We claim that if $\lim_{n\to\infty}\int_{d(n,k)}\|f(t)\|^p\,d\mu = 0$ for all $k \in \mathbf{N}$, then (f_n) converges to f in norm. Given $\varepsilon > 0$, there is a $k \in \mathbf{N}$ such that $1/k < \varepsilon/2$. Since $\lim_{n\to\infty}\int_{d(n,k)}\|f(t)\|^p\,d\mu = 0$, there exists N such that, if $n > N$, then

$$\int_{d(n,k)} \|f(t)\|^p\,d\mu < \varepsilon^p/4^p.$$

Hence, if $n > N$, then

$$\begin{aligned}
&\int \|f(t) - f_n(t)\|^p\,d\mu \\
&= \int_{d(n,k)} \|f(t) - f_n(t)\|^p\,d\mu + \int_{\Omega - d(n,k)} \|f(t) - f_n(t)\|^p\,d\mu \\
&\leqslant 2^p \int_{d(n,k)} \|f(t)\|^p\,d\mu + \int_{\Omega - d(n,k)} \|f(t)\|^p/k^p\,d\mu \\
&< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p.
\end{aligned}$$

Now, suppose that (f_n) does not converge to f in norm; then there exist $\varepsilon > 0$ and $k \in \mathbf{N}$ such that

$$\int_{d(n,k)} \|f(t)\|^p\,d\mu > \varepsilon \quad \text{for infinitely many } n.$$

By passing to a subsequence, we may assume that

$$\int_{d(n,k)} \|f(t)\|^p\,d\mu > \varepsilon \quad \text{for all } n.$$

Since (f_n) converges to f weakly, for each $m \in \mathbf{N}$, there is $(a_i^m)_{i=1}^{N_m}$ such that $\sum_{i=1}^{N_m} a_i^m = 1$, $a_i^m \geqslant 0$ and $\|\sum_{i=1}^{N_m} a_i^m f_i - f\| < 1/m$. By passing to a sequence of $(\sum_{i=1}^{N_m} a_i^m f_i)_{m=1}^{\infty}$, we may assume that $(\sum_{i=1}^{N_m} a_i^m f_i)$ converges to f a.e. Let λ be the probability given by

$$\lambda(A) = \int_A \|f(t)\|^p\,d\mu \quad \text{for all } A \in \Sigma.$$

Then $\lambda(d(n,k)) > \varepsilon$. For each m, let

$$S(m) = \left\{ t\colon \left(\sum_{i=1}^{N_m} a_i^m x_{d(i,k)} \right)(t) > \frac{\varepsilon}{2} \right\}.$$

Since $\int \sum_{i=1}^{N_m} a_i^m x_{d(i,k)}(t)\,d\lambda > \varepsilon$ and λ is a probability measure, $\lambda(S(m)) > \varepsilon/2$. Let $S = \{t\colon t \in S(m) \text{ for infinitely many } m\}$. Then $\lambda(S) \neq 0$ and there is $t' \in S$ such that $(\sum_{i=1}^{N_m} a_i^m f_i(t'))_{m=1}^{\infty}$ converges to $f(t')$ in norm. Let $T = \{n\colon t' \in d(n,k)\}$. If $t' \in S(m)$, then $\sum_{i\in T} a_i^m > \varepsilon/2$ and

$$\begin{aligned}
\sum_{i=1}^{N_m} a_i^m f_i(t') &= \sum_{i\in T} a_i^m f_i(t') + \sum_{i\notin T} a_i^m f_i(t') \\
&= \left(\sum_{i\in T} a_i^m \right)\left(\sum_{i\in T} a_i^m f_i(t') \Big/ \sum_{i\in T} a_i^m \right) \\
&\quad + \left(\sum_{i\notin T} a_i^m \right)\left(\sum_{i\notin T} a_i^m f_i(t') \Big/ \sum_{i\notin T} a_i^m \right).
\end{aligned}$$

Since $\|f_i(t') - f(t')\| > \|f(t')\|/k$, $\|f_i(t')\| = \|f(t')\|$ for all $i \in T$ and X has the property (G), there is $\delta > 0$ such that if $x \in \overline{\mathrm{co}}\{f_i(t') | i \in T\}$ then $\|x - f(t')\| > \delta$. Since $t' \in S(m)$ infinitely often, there exist sequences (x_n), (y_n) and (α_n) such that $\|x_n - f(t')\| > \delta$ for all n, $1 \geqslant \alpha_n > \varepsilon/2$ and

$$\lim_{n \to \infty} \alpha_n x_n + (1 - \alpha_n) y_n = f(t').$$

This contradicts the fact that X is (MLUR). So f_n must converge to f in norm. Q.E.D.

4. Since a Banach space X contains a copy of l_1 if and only if $L^p(\mu, X)$, $1 < p < \infty$, contains a copy of l_1 [7], we have the following theorem.

THEOREM 3. *If l_1 is not isomorphic to a subspace of X, then the following are equivalent*:

(i) *X has the property* (HR),

(ii) *X has the property* (G),

(iii) $L^p(\mu, X)$, $1 < p < \infty$, *has the property* (HR),

(iv) $L^p(\mu, X)$, $1 < p < \infty$, *has the property* (G).

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