ON A 2-DIMENSIONAL EINSTEIN KAEHLER SUBMANIFOLD OF A COMPLEX SPACE FORM

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ABSTRACT. In this paper we consider when a Kaehler submanifold of a complex space form is Einstein with respect to the induced metric. Then we shall show that (1) a 2-dimensional complete Kaehler submanifold M of a 4-dimensional complex projective space $P^4(C)$ is Einstein if and only if M is holomorphically isometric to $P^2(C)$ which is totally geodesic in $P^4(C)$ or a hyperquadric $Q^2(C)$ in $P^3(C)$ which is totally geodesic in $P^4(C)$, and that (2) if M is a 2-dimensional Einstein Kaehler submanifold of a 4-dimensional complex space form $\tilde{M}^4(\tilde{c})$ of nonpositive constant holomorphic sectional curvature $\tilde{c}$, then M is totally geodesic.

Let M^n be an n -dimensional Kaehler submanifold of an $(n+p)$ -dimensional complex space form $\tilde{M}^{n+p}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. Let $P^{n+p}(C)$ be an $(n+p)$ -dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let C^{n+p} (or D^{n+p}) be a complex $(n+p)$ -dimensional Euclidean space with the usual Hermitian metric (resp. the $(n+p)$ -dimensional open unit ball in C^{n+p} with the natural complex structure and the Bergman metric of constant holomorphic sectional curvature -1). Any m -dimensional complex space form is (after multiplying the metric by a suitable constant) locally complex analytically isometric to $P^m(C)$, C^m or D^m , according as the holomorphic sectional curvature is positive, zero or negative. We consider the following question:

When is M Einstein with respect to the induced metric?

The answer is known in the following cases:

(1) $p = 1$ (Smyth [8]).

A complete hypersurface M of $P^{n+1}(C)$ is Einstein if and only if M is holomorphically isometric to $P^n(C)$ or a complex hyperquadric $Q^n(C)$.

A complete and simply connected hypersurface M of C^{n+1} (resp. D^{n+1}) is Einstein if and only if M is holomorphically isometric to C^n (resp. D^n).

(2) A complete intersection submanifold (Hano [3]).

A complete intersection submanifold M^n of $P^{n+p}(C)$ is Einstein if and only if M is $P^n(C)$ which is totally geodesic in $P^{n+p}(C)$ or a hyperquadric $Q^n(C)$ in $P^{n+1}(C)$ which is totally geodesic in $P^{n+p}(C)$.

Received by the editors April 2, 1983 and, in revised form, February 22, 1984.

1980 *Mathematics Subject Classification*. Primary 53B25, 53C40.

Key words and phrases. Einstein Kaehler submanifolds, complex space forms, second fundamental forms.

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In this paper we consider the case of $n = 2$. We show

THEOREM A. *A complete Kaehler submanifold M^2 of $P^4(C)$ is Einstein if and only if M is holomorphically isometric to $P^2(C)$ which is totally geodesic in $P^4(C)$ or a hyperquadric $Q^2(C)$ in $P^3(C)$ which is totally geodesic in $P^4(C)$.*

Since for each $n \geq 1$ there exists a Kaehler imbedding $f: P_{1/2}^n(C) \rightarrow P^{n+(n(n+1))/2}(C)$ (O'Neill [7]), the codimension of Theorem A is best possible, where $1/2$ of $P_{1/2}^n(C)$ denotes constant holomorphic sectional curvature.

THEOREM B. *Let M^2 be an Einstein Kaehler submanifold immersed in $\tilde{M}^4(\tilde{c})$, $\tilde{c} \leq 0$. Then M is totally geodesic.*

1. Kaehler submanifolds. As in the introduction, let M^n be a Kaehler submanifold of $\tilde{M}^{n+p}(\tilde{c})$. And let f be its Kaehler immersion (i.e., holomorphically isometric immersion). Let J_0 (resp. J) be the complex structure of M (resp. $\tilde{M}$) and let g_0 (resp. g) be the Kaehler metric of M (resp. $\tilde{M}$). In order to simplify the presentation we identify, for each $x \in M$, the tangent space $T_x(M)$ with $f_*(T_x(M)) \subset T_{f(x)}(\tilde{M})$ by means of f_* . The normal space $T_x^\perp(M)$ is the subspace of $T_{f(x)}(\tilde{M})$ consisting of all $X \in T_{f(x)}(\tilde{M})$ which are orthogonal, with respect to g , to the subspace $f_*(T_x(M))$. Since $f^*g = g_0$ and $J \circ f_* = f_* \circ J_0$, where J_0 is the almost complex structure of M , the structures g_0 and J_0 on $T_x(M)$ are respectively identified with the restrictions of the structures g and J to the subspace $f_*(T_x(M))$. With this identification in mind we drop the subscript 0 on g_0 and J_0 . Let ∇ (resp. $\tilde{\nabla}$) denote the covariant differentiation in M (resp. $\tilde{M}$), and let $\nabla^\perp$ denote covariant differentiation in the normal bundle.

With each $\xi \in T_x^\perp(M)$ is associated a linear transformation of $T_x(M)$ in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x(M)$. $A_\xi X$ depends only on ξ at x and X . Given an orthonormal basis $\xi_1, \dots, \xi_p$, $\xi_{1^*} = J\xi_1, \dots, \xi_{p^*} = J\xi_p$ of $T_x^\perp(M)$. We write $A_\lambda = A_{\xi_\lambda}$ and call the A_λ 's the second fundamental forms associated with $\xi_1, \dots, \xi_p$, $\xi_{1^*}, \dots, \xi_{p^*}$ are now orthonormal normal vector field in a neighborhood U of x , they determine normal connection forms $s_{\alpha\beta}, t_{\alpha\beta}$ in U by

$$(1.1) \quad \nabla_X^\perp \xi_\alpha = \sum_\beta s_{\beta\alpha}(X) \xi_\beta + \sum_\beta t_{\beta\alpha}(X) \xi_{\beta^*}, \quad s_{\alpha\beta} + s_{\beta\alpha} = 0, t_{\alpha\beta} - t_{\beta\alpha} = 0$$

for $X \in T_x(M)$. Let S and ρ be the Ricci tensor and the scalar curvature for M , respectively. Then we have the following relationship (in this paper α, β, γ run from 1 to p and λ, μ run from $1, \dots, p, 1^*, \dots, p^*$, except when noted) [5]:

$$(1.2) \quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$

$$(1.3) \quad \sigma(X, Y) = \sum g(A_\lambda X, Y) \xi_\lambda,$$

$$(1.4) \quad g(A_\alpha X, Y) = g(A_\alpha Y, X), \quad A_{\alpha^*} = JA_\alpha, \quad A_\alpha J = -A_\alpha J,$$

$$(1.5) \quad (\nabla_X A_\alpha) Y - \sum_\beta s_{\beta\alpha}(X) A_\beta Y - \sum_\beta t_{\beta\alpha}(X) JA_\beta Y \\ = (\nabla_Y A_\alpha) X - \sum_\beta s_{\beta\alpha}(Y) A_\beta X - \sum_\beta t_{\beta\alpha}(Y) JA_\beta X - \text{Codazzi equation},$$

$$(1.6) \quad S = \frac{n+1}{2} \tilde{c}I - 2 \sum_{\alpha} A_{\alpha}^2,$$

$$(1.7) \quad \rho = n(n+1)\tilde{c} - \|\sigma\|^2,$$

where X, Y are tangent to M , σ is also called the second fundamental form of f , and $I, \|\sigma\|$ denote the identity transformation of $T_x(M)$, the length of σ , so that

$$\|\sigma\|^2 = 2 \sum \text{trace } A_{\alpha}^2.$$

2. Proposition. Let M^n be a Kaehler submanifold of $\tilde{M}^{n+p}(\tilde{c})$.

DEFINITION. For $x \in M$, the first normal space, $N_1(x)$, is the orthogonal complement in $T_x(M)$ of the set $N_0(x) = \{\xi \in T_x(M) | A_{\xi} = 0\}$.

PROPOSITION 1. Let M^2 be a 2-dimensional Einstein Kaehler submanifold of $\tilde{M}^{2+p}(\tilde{c})$. Then $\dim N_1(x) \neq 2$.

PROOF. We assume that $\dim N_1(x) = 2$. Then $\dim N_1(y) \geq 2$ in a neighborhood U_0 of x . In terms of codimension $M = 2$ we have $\dim N_1(y) = 2$ in U_0 . Choose local orthonormal vector fields $\xi_1, \xi_2, J\xi_1, J\xi_2$ so that a symmetric $(4, 4)$ -matrix (trace $A_{\lambda}A_{\mu}$) can be diagonalized.

Let $\Omega = \{e_1, e_2, e_{1*} = Je_1, e_{2*} = Je_2\}$ be an orthonormal basis for $T_x(M)$ with respect to which A_1 is diagonal of the form

$$A_1 = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & -\alpha & \\ & & & -\beta \end{pmatrix}.$$

By means of

$$\begin{aligned} g(A_2e_1, e_{2*}) &= g(A_2e_2, e_{1*}), \\ g(A_2e_{a*}, e_{b*}) &= -g(A_2e_a, e_b), \quad a, b = 1, 2, \end{aligned}$$

A_2 is represented by the matrix

$$A_2 = \begin{pmatrix} a_{11} & a_{12} & a_{11*} & a_{12*} \\ a_{12} & a_{22} & a_{12*} & a_{22*} \\ a_{11*} & a_{12*} & -a_{11} & -a_{12} \\ a_{12*} & a_{22*} & -a_{12} & -a_{22} \end{pmatrix}.$$

Since M^2 is Einstein, i.e., $S = (\rho/(2 \dim M))I$, we have

$$(2.1) \quad A_1^2 + A_2^2 = \frac{1}{8} \|\sigma\|^2 I.$$

From $\text{trace } A_1 A_2 = \text{trace } A_1 A_{2*} = 0$ we have

$$(2.2) \quad a_{11}\alpha + a_{22}\beta = 0,$$

$$(2.3) \quad a_{11*}\alpha + a_{22*}\beta = 0.$$

Case 1. $a_{11} = a_{22} = a_{11*} = a_{22*} = 0$. Then $A_2^2 = (a_{12}^2 + a_{12*}^2)I$. Thus from (2.1) we obtain $\alpha^2 = \beta^2$, i.e., $A_1^2 = \alpha^2 I$. Then

$$A_1 \circ A_{\xi_1 + \xi_2}^2 = A_{\xi_1 + \xi_2}^2 \circ A_1,$$

$$A_1 \circ A_{\xi_1 + J\xi_2}^2 = A_{\xi_1 + J\xi_2}^2 \circ A_1,$$

$$A_{\xi_1 + \xi_2}^2 \circ A_{\xi_1 + J\xi_2}^2 = A_{\xi_1 + J\xi_2}^2 \circ A_{\xi_1 + \xi_2}^2.$$

Hence A_1 , $A_{\xi_1 + \xi_2}^2$ and $A_{\xi_1 + J\xi_2}^2$ can be simultaneously diagonalized. We may assume that from the beginning we choose such a basis Ω . Then we obtain

$$A_{\xi_1 + \xi_2}^2 - A_1^2 - A_2^2 = A_1 A_2 + A_2 A_1$$

$$= \begin{pmatrix} 0 & a_{12}\alpha + a_{12}\beta & 0 & a_{12*}\alpha - a_{12*}\beta \\ a_{12}\alpha + a_{12}\beta & 0 & -a_{12*}\alpha + a_{12*}\beta & 0 \\ 0 & -a_{12*}\alpha + a_{12*}\beta & 0 & a_{12}\alpha + a_{12}\beta \\ a_{12*}\alpha - a_{12*}\beta & 0 & a_{12}\alpha + a_{12}\beta & 0 \end{pmatrix},$$

$$A_{\xi_1 + J\xi_2}^2 - A_1^2 - A_2^2$$

$$= \begin{pmatrix} 0 & -a_{12*}\alpha - a_{12*}\beta & 0 & a_{12}\alpha - a_{12}\beta \\ -a_{12*}\alpha - a_{12*}\beta & 0 & -a_{12}\alpha + a_{12}\beta & 0 \\ 0 & -a_{12}\alpha + a_{12}\beta & 0 & -a_{12*}\alpha - a_{12*}\beta \\ a_{12}\alpha - a_{12}\beta & 0 & -a_{12*}\alpha - a_{12*}\beta & 0 \end{pmatrix}.$$

Therefore

$$a_{12}\alpha + a_{12}\beta = 0, \quad a_{12}\alpha - a_{12}\beta = 0,$$

$$a_{12*}\alpha + a_{12*}\beta = 0, \quad a_{12*}\alpha - a_{12*}\beta = 0.$$

Since we may assume that $(\alpha \beta) \neq 0$, we obtain $a_{12} = a_{12*} = 0$. This contradicts the assumption that $k = 2$.

Case 2. $(a_{11} a_{22} a_{11*} a_{22*}) \neq 0$. Since A_2^2 is diagonal, we obtain

$$(a_{11} + a_{22})a_{12} + (a_{11*} + a_{22*})a_{12*} = 0,$$

$$(a_{22*} - a_{11*})a_{12} + (a_{11} - a_{22})a_{12*} = 0.$$

If $(a_{12} a_{12*}) \neq 0$, i.e., $a_{11}^2 + a_{11*}^2 = a_{22}^2 + a_{22*}^2$, then we can put $A_2^2 = a^2 I$, where

$$a^2 = a_{11}^2 + a_{12}^2 + a_{11*}^2 + a_{12*}^2 = a_{22}^2 + a_{12}^2 + a_{22*}^2 + a_{12*}^2.$$

Combining this fact with (2.1), we have $\alpha^2 = \beta^2$, i.e., $A_1^2 = \alpha^2 I$. Then we have

$$A_1 \circ A_{\xi_1 + \xi_2}^2 = A_{\xi_1 + \xi_2}^2 \circ A_1,$$

$$A_1 \circ A_{\xi_1 + J\xi_2}^2 = A_{\xi_1 + J\xi_2}^2 \circ A_1,$$

$$A_{\xi_1 + \xi_2}^2 \circ A_{\xi_1 + J\xi_2}^2 = A_{\xi_1 + J\xi_2}^2 \circ A_{\xi_1 + \xi_2}^2.$$

Thus by the same argument as Case 1 we know that the case of $(a_{12} a_{12*}) \neq 0$ cannot occur. Thus we may assume that $a_{12} = a_{12*} = 0$. Then from (2.1) we have

$$(2.4) \quad \alpha^2 + a_{11}^2 + a_{11*}^2 = \frac{1}{8} \|\sigma\|^2,$$

$$(2.5) \quad \beta^2 + a_{22}^2 + a_{22*}^2 = \frac{1}{8} \|\sigma\|^2.$$

Multiplying (2.4) (resp. (2.5)), by α^2 (resp. β^2) we have

$$(2.6) \quad \alpha^4 + a_{11}^2 \alpha^2 + a_{11*}^2 \alpha^2 = \frac{1}{8} \alpha^2 \|\sigma\|^2,$$

$$(2.7) \quad \beta^4 + a_{22}^2 \beta^2 + a_{22*}^2 \beta^2 = \frac{1}{8} \beta^2 \|\sigma\|^2.$$

Using (2.2) and (2.3), we get from (2.6) that

$$(2.6)' \quad \alpha^4 + a_{22}^2 \beta^2 + a_{22*}^2 \beta^2 = \frac{1}{8} \alpha^2 \|\sigma\|^2.$$

Subtracting (2.7) from (2.6)', we obtain

$$(\alpha^2 - \beta^2)(\alpha^2 + \beta^2 - \frac{1}{8} \|\sigma\|^2) = 0.$$

If $\alpha^2 \neq \beta^2$, then from (2.4) and (2.5) we have

$$A_2^2 = \begin{pmatrix} \beta^2 & & & \\ & \alpha^2 & & \\ & & \beta^2 & \\ & & & \alpha^2 \end{pmatrix}.$$

Hence we know that $\det A_1^2 = \det A_2^2$. Since we have

$$\det A_1 = \alpha^2 \beta^2, \quad \det A_2 = a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2 + a_{11*}^2 a_{22}^2 + a_{11*}^2 a_{22*}^2,$$

we obtain

$$(2.8) \quad \alpha^2 \beta^2 = a_{11}^2 a_{22}^2 + a_{11}^2 a_{22*}^2 + a_{11*}^2 a_{22}^2 + a_{11*}^2 a_{22*}^2.$$

On the other hand, from (2.2) and (2.3) we have

$$(2.9) \quad a_{11} a_{22*} - a_{11*} a_{22} = 0,$$

since we may assume that $(\alpha \beta) \neq 0$. Combining (2.8) and (2.9), we obtain

$$(2.10) \quad \alpha^2 \beta^2 = (a_{11} a_{22} + a_{11*} a_{22*})^2.$$

If $\alpha \beta = a_{11} a_{22} + a_{11*} a_{22*}$, then multiplying (2.2) by α and using (2.9), we have

$$a_{11}(a_{22}^2 + a_{22*}^2) = 0; \quad \text{i.e.,} \quad a_{11} \alpha^2 = 0.$$

Similarly, we have

$$\begin{aligned} a_{22}(a_{11}^2 + a_{11*}^2) &= 0; \quad \text{i.e.,} \quad a_{22} \beta^2 = 0, \\ a_{11*}(a_{22}^2 + a_{22*}^2) &= 0; \quad \text{i.e.,} \quad a_{11*} \alpha^2 = 0, \\ a_{22*}(a_{11}^2 + a_{11*}^2) &= 0; \quad \text{i.e.,} \quad a_{22*} \beta^2 = 0. \end{aligned}$$

Hence, we obtain

$$\begin{pmatrix} \alpha^2 & & & \\ & \beta^2 & & \\ & & \alpha^2 & \\ & & & \beta^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{22} \\ a_{11*} \\ a_{22*} \end{pmatrix} = 0.$$

Since $(a_{11} a_{22} a_{11*} a_{22*}) \neq 0$, we have $\alpha \beta = 0$. Assume that $\alpha \beta \neq 0$. Then from (2.10) we have $\alpha \beta + a_{11} a_{22} + a_{11*} a_{22*} = 0$ and

$$\begin{aligned} \sigma(e_1, e_1) &= \alpha \xi_1 + a_{11} \xi_2 - a_{11*} \xi_{2*}, \\ \sigma(e_2, e_2) &= \beta \xi_1 + a_{22} \xi_2 - a_{22*} \xi_{2*}. \end{aligned}$$

Hence, we obtain

$$g(\sigma(e_1, e_1), \sigma(e_2, e_2)) = g(\sigma(e_1, e_1), \sigma(e_2, e_{2*})) = 0$$

and $\sigma(e_1, e_2) = 0$. On the other hand, from (2.1) we have

$$(2.11) \quad \|\sigma(E_a, E_a)\|^2 + \|\sigma(E_1, E_2)\|^2 = \frac{1}{8} \|\sigma\|^2, \quad a = 1, 2,$$

$$(2.12) \quad g(\sigma(E_1, E_1), \sigma(E_1, E_2)) + g(\sigma(E_2, E_2), \sigma(E_1, E_2)) = 0,$$

$$(2.13) \quad g(\sigma(E_1, E_1), \sigma(E_1, E_{2*})) - g(\sigma(E_2, E_2), \sigma(E_1, E_{2*})) = 0,$$

where $\{E_1, E_2, E_{1*}, E_{2*}\}$ is a suitable local orthonormal frame around x such that $E_j = e_j$ at $x, j = 1, 2, 1*, 2*$. Then from (2.11) we have

$$g(\nabla_X^\perp(\sigma(E_a, E_a)), \sigma(E_a, E_a)) + g(\nabla_X^\perp(\sigma(E_1, E_2)), \sigma(E_1, E_2)) = 0,$$

i.e.,

$$g(\nabla_X^\perp(\sigma(E_a, E_a)), \sigma(e_a, e_a)) = 0 \quad \text{at } x.$$

Since $\sigma(e_1, e_2) = 0$ at x , we have [5]

$$\begin{aligned} (\nabla_X' \sigma)(e_a, e_a) &= \nabla_X^\perp(\sigma(E_a, E_a)) - \sigma(\nabla_X E_a, E_a) - \sigma(E_a, \nabla_X E_a) \\ &= \nabla_X^\perp(\sigma(E_a, E_a)) - 2g(\nabla_X E_a, E_{a*})\sigma(e_a, e_{a*}), \end{aligned}$$

where ∇' denotes covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle). Hence, we obtain

$$(2.14) \quad g((\nabla_X' \sigma)(e_a, e_a), \sigma(e_a, e_a)) = 0.$$

From (2.12) and $\sigma(e_1, e_2) = 0$ at x we have

$$g(\nabla_X^\perp(\sigma(E_1, E_2)), \sigma(e_1, e_1)) + g(\nabla_X^\perp(\sigma(E_1, E_2)), \sigma(e_2, e_2)) = 0.$$

Using (2.11) and

$$\begin{aligned} (2.15) \quad (\nabla_X' \sigma)(e_1, e_2) &= \nabla_X^\perp(\sigma(E_1, E_2)) - g(\nabla_X E_1, E_2)\sigma(e_2, e_2) \\ &\quad - g(\nabla_X E_1, E_{2*})\sigma(e_2, e_{2*}) - g(\nabla_X E_2, E_1)\sigma(e_1, e_1) \\ &\quad - g(\nabla_X E_2, E_{1*})\sigma(e_1, e_{1*}), \end{aligned}$$

we obtain

$$\begin{aligned} &g((\nabla_X' \sigma)(e_1, e_2), \sigma(e_1, e_1)) + g(\nabla_X E_2, E_1)\|\sigma(e_1, e_1)\|^2 \\ &\quad + g((\nabla_X' \sigma)(e_1, e_2), \sigma(e_2, e_2)) + g(\nabla_X E_1, E_2)\|\sigma(e_2, e_2)\|^2 = 0, \end{aligned}$$

i.e.,

$$g((\nabla_X' \sigma)(e_1, e_2), \sigma(e_1, e_1)) + g((\nabla_X' \sigma)(e_1, e_2), \sigma(e_2, e_2)) = 0.$$

Since from (2.14) we obtain

$$g((\nabla_{e_1}' \sigma)(e_1, e_2), \sigma(e_1, e_1)) = g((\nabla_{e_2}' \sigma)(e_1, e_1), \sigma(e_1, e_1)) = 0,$$

we have

$$g((\nabla_{e_2}' \sigma)(e_1, e_1), \sigma(e_2, e_2)) = g((\nabla_{e_1}' \sigma)(e_1, e_2), \sigma(e_2, e_2)) = 0.$$

Similarly, from (2.13) we have

$$g((\nabla_{e_{2*}}' \sigma)(e_1, e_1), \sigma(e_2, e_2)) = 0.$$

Hence,

$$g((\nabla_{e_2}' \sigma)(e_1, e_1), \sigma(e_1, e_{1*})) = g((\nabla_{e_2}' \sigma)(e_1, e_1), \sigma(e_2, e_{2*})) = 0.$$

Thus

$$(2.16) \quad (\nabla'_{e_2} \sigma)(e_1, e_1) = 0 \quad \text{at } x.$$

Similarly, we obtain

$$(2.17) \quad (\nabla'_{e_1} \sigma)(e_2, e_2) = 0 \quad \text{at } x.$$

Since $\alpha\beta(\alpha^2 - \beta^2) \neq 0$, $\alpha\beta(\alpha^2 - \beta^2) \neq 0$ in a neighborhood U of x . Using the minimal polynomial of A_1 , we can show that $\{X|A_1X = \gamma X\}$ is differentiable (e.g. [4]), where γ is the eigenvalue of A_1 . Thus we may assume that A_1 is diagonalized with respect to $\{E_1, E_2, E_1^*, E_2^*\}$ in U . Then we have

$$g(\sigma(E_1, E_1), \sigma(E_2, E_2)) = 0.$$

Thus

$$g(\nabla_X^\perp(\sigma(E_1, E_1)), \sigma(E_2, E_2)) + g(\sigma(E_1, E_1), \nabla_X^\perp(\sigma(E_2, E_2))) = 0.$$

Hence,

$$\begin{aligned} g((\nabla'_{e_1} \sigma)(e_1, e_1), \sigma(e_2, e_2)) &= 0, \\ g((\nabla'_{e_2} \sigma)(e_2, e_2), \sigma(e_1, e_1)) &= 0 \quad \text{at } x. \end{aligned}$$

Therefore we know that $\nabla' \sigma = 0$ at x . On the other hand, from the equation [6]

$$\frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 - 8 \operatorname{trace} \left(\sum A_\alpha^2 \right)^2 - \sum (\operatorname{trace} A_\lambda A_\mu)^2 + 2\tilde{c} \|\sigma\|^2,$$

we obtain $\|\sigma\|^2 = 0$ [5]. This is a contradiction. We know that the case $\alpha\beta \neq 0$ and $\alpha\beta + a_{11}a_{22} + a_{11^*}a_{22^*} = 0$ cannot occur. By the similar argument with the above case we know that both cases $\alpha\beta = 0$ and $\alpha^2 = \beta^2$ cannot occur. This completes the proof.

3. Proofs of Theorems A and B. From Proposition 1 we know that $k \leq 1$.

PROPOSITION 2. $N_1(x)$ has constant dimension on M^2 .

PROOF. If the second fundamental form $\sigma(X, Y) = 0$ for all $x \in M^2$, then $N_1(x)$ has constant dimension 0, and the proofs are complete.

Suppose that $\sigma(X, Y) \neq 0$ at $x_0 \in M^2$. Let U be a neighborhood of x_0 on which $\sigma(X, Y) \neq 0$. Then $N_1(u)$, $u \in U$, has constant dimension 1 on U . Since M^2 is Einstein,

$$A_1 = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & -\alpha & \\ & & & -\alpha \end{pmatrix},$$

where $\alpha = (1/2\sqrt{2})\|\sigma\|$. This implies α is constant on U . Consider the set S defined by

$$S = \{x \in M^2 | \alpha(x) = \alpha(x_0)\}.$$

Since α is continuous on M^2 , we know S is closed. However, the above argument

implies S is open. Since $x_0 \in S$, we know $S \neq \emptyset$; so the connectedness of M^2 implies $S = M^2$. Hence, $\alpha = \alpha(x_0)$ on M^2 , and $N_1(x)$ has constant dimension 1 on M^2 .

In the case where $N_1(x)$ has constant dimension 0, M^2 is totally geodesic. To complete the proofs of Theorems A and B, we must show that when $N_1(x)$ has constant dimension 1, we can reduce to codimension 1.

Let U be any coordinate neighborhood of M^2 . We choose orthonormal normal vector fields ξ_1, ξ_2 on U so that $\xi_1, \xi_2, J\xi_1, J\xi_2$ span $T_u^\perp(M^2)$ for any $u \in U$ and such that $\xi_1, J\xi_1$ span $N_1(u)$ for any $u \in U$. We then prove

PROPOSITION 3. *For any $x \in U$ and $X \in T_x(M^2)$ the following equations are true:*

- (i) $\nabla_X^\perp \xi_1 = t_{11}(X)J\xi_1$,
- (ii) $\nabla_X^\perp \xi_2$ and $\nabla_X^\perp J\xi_2 \in \text{span}\{\xi_2, J\xi_2\}$.

PROOF. Equation (1.5) says that

$$(\nabla_X A_2)Y - \sum_{\beta=1}^2 s_{\beta 2}(X)A_\beta Y - \sum_{\beta=1}^2 t_{\beta 2}(X)JA_\beta Y$$

is symmetric in X and Y .

Since $A_2 = 0$, $\nabla_X A_2 = 0$ and (1.5) can be written as

$$(3.1) \quad s_{12}(X)A_1 Y + t_{12}(X)JA_1 Y = s_{12}(Y)A_1 X + t_{12}(Y)JA_1 X.$$

We can choose X, Y linearly independent vectors so that $A_1 X = \alpha X$ and $A_1 Y = \alpha Y$, since $\dim M = 2$. Then (3.1) becomes

$$(3.2) \quad s_{12}(X)\alpha Y + t_{12}(X)\alpha JY = s_{12}(Y)\alpha X + t_{12}(Y)\alpha JX.$$

But X, Y, JX, JY are linearly independent, so (3.2) implies

$$(3.3) \quad s_{12}(X) = t_{12}(X) = 0.$$

A similar calculation shows that (3.3) holds for a vector X , so that $A_1 X = -\alpha X$, and, hence, (3.3) holds for all $X \in T_x(M^2)$. We recall that

$$(1.1) \quad \nabla_X^\perp \xi_\beta = \sum_{\gamma=1}^2 s_{\gamma\beta}(X)\xi_\gamma + \sum_{\gamma=1}^2 t_{\gamma\beta}(X)J\xi_\gamma, \quad \beta = 1, 2.$$

Then $s_{\beta\gamma} = -s_{\gamma\beta}$ and $t_{\beta\gamma} = t_{\gamma\beta}$ and (3.3) imply that for $\beta = 1$, (1.1) becomes

$$\nabla_X^\perp \xi_1 = t_{11}(X)J\xi_1,$$

proving (i). For the same reasons, for $\beta = 2$, (1.1) becomes

$$(3.4) \quad \nabla_X^\perp \xi_2 = t_{22}(X)J\xi_2.$$

Then $\nabla_X^\perp J\xi_2 = J(\nabla_X^\perp \xi_2)$ and (3.4) proves (ii).

Proposition 3 implies $N_1(x)$ and $N_0(x)$ are invariant with respect to $\nabla^\perp$. Propositions 2 and 3 and the following lemmas complete the proofs of Theorems A and B.

LEMMA 4 (CECIL [1]). *Let $f: M^n \rightarrow \tilde{M}^{n+p}(\tilde{c})$ be a Kaehler immersion of M^n into $\tilde{M}^{n+p}(\tilde{c})$. Suppose $N_1(x)$ has constant dimension k and is parallel with respect to the normal connection. Then there is a totally geodesic $(n+k)$ -dimensional submanifold $\tilde{M}^{n+k}(\tilde{c})$, such that $f(M^n) \subset \tilde{M}^{n+k}(\tilde{c})$.*

LEMMA 5 (CHERN [2] AND SMYTH [8]). *Let M^n be an Einstein Kaehler hypersurface of $\tilde{M}^{n+1}(\tilde{c})$. If $n \geq 2$, then M is totally geodesic in $\tilde{M}$ or $S = (n/2)\tilde{c}g$, the latter case arising only when $\tilde{c} > 0$. Moreover, the immersion is rigid.*

PROOF. See, for example, [6, p. 95].

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