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FINITELY ADDITIVE SUPERMARTINGALES ARE DIFFERENCES OF MARTINGALES AND ADAPTED INCREASING PROCESSES

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ABSTRACT. It is shown that any nonnegative bounded supermartingale admits a Doob-Meyer decomposition as a difference of a martingale and an adapted increasing process upon appropriate choice of a reference probability measure which may be only finitely additive.

Introduction. In [Armstrong, **1983**] it is shown that every bounded finitely additive supermartingale is a decreasing process with respect to some reference probability measure P. This concept of a decreasing (or increasing) process is weaker than that corresponding to decreasing (increasing) processes of random variables adapted to a filtration. The corresponding class of finitely additive processes are called adapted decreasing (increasing) processes with respect to P. Theorem B asserts that for every bounded nonnegative finitely additive supermartingale Y there is a probability P so that $Y = M - A$ where M is a martingale and A is an adapted increasing process with respect to P. In order to establish this it is necessary in Proposition A to show that for $g = \{g_t\colon t \in T\}$ an ordinary L^1-bounded nonnegative supermartingale adapted to the linearly ordered filtration $\{\mathscr{F}_t\colon t \in T\}$ on the probability measure space $(X, \mathscr{F}, P)$ to be expressed as the difference $m - a$ where m is a martingale and a is an increasing process with $0 = \inf_t a_t$ it is necessary and sufficient that $\{g_\tau\colon \tau$ simple T-valued stopping time $\leqslant t\}$ be uniformly integrable for all $t \in T$. This extends the usual Doob-Meyer Decomposition Theorem in allowing arbitrary linearly ordered T.

Finitely additive supermartingales are differences of martingales and increasing adapted sequences. The Doob-Meyer Decomposition Theorem asserts that a nonnegative L^1-bounded supermartingale $f = \{f_t\colon 0 \leqslant t < \infty\}$ adapted to a filtration $\{\mathscr{F}_t\colon 0 \leqslant t < \infty\}$ of sub-σ-algebras in a probability space $(X, \mathscr{F}, P)$ admits a decomposition $f = m - a$ where m is a martingale and a is an increasing process with $a_0 = 0$ iff f is of class DL. We recall that f is of class DL iff $\{f_\tau\colon \tau$ stopping time $\leqslant t\}$ is uniformly integrable (i.e., $\sigma(L^1, L^\infty)$-relatively compact) for each $t \in [0, \infty)$. Attention may be confined to stopping times τ with only finitely many

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values (i.e., simple stopping times) in the definition of class DL. Usually the Doob-Meyer Decomposition Theorem is stated for right-continuous f but holds for arbitrary f as is shown in Theorem 20 of Appendix I of Dellacherie and Meyer [**1982**]. See Mertens [**1971**] for one of the first results in this vein. Imposing predictability guarantees uniqueness on the increasing process but this will not be of concern to us here.

In Armstrong [**1983**] finitely additive supermartingales were introduced. See Gut and Schmidt [**1983**] for Schmidt's work on finitely additive supermartingales on the integers. The topic of concern here is the Doob-Meyer Decomposition Theorem in this context. We recall that in Armstrong [**1983**] filtrations of σ-algebras are replaced by linearly ordered families of subalgebras of a Boolean algebra $\mathscr{B}$. If Γ is a chain of subalgebras of $\mathscr{B}$ then a process Y on Γ is a family $\{Y_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ where each $Y_{\mathscr{A}}$ is a finitely additive measure of bounded variation on the subalgebra $\mathscr{A}$ of $\mathscr{B}$. Suppose that $\mathscr{B}$ is the σ-algebra $\mathscr{F}$ of a probability space $(X, \mathscr{F}, P)$, Γ is a filtration $\{\mathscr{F}_t\colon 0 \leqslant t < \infty\}$, and that $f = \{f_t\colon 0 \leqslant t < \infty\}$ is an L^1-bounded stochastic process adapted to Γ. In this case one obtains a process Y on Γ by setting Y_t equal to that measure on $\mathscr{F}_t$ with P-density f_t. Note that $E(f_s|\mathscr{F}_t)$ corresponds to restricting Y_s to $\mathscr{F}_t$. As a result we adopt the conditional expectation notation $E(\mu|\mathscr{A})$ for the restriction to $\mathscr{A}$ of a finitely additive μ on a superalgebra of $\mathscr{A}$. One important distinction between processes of finitely additive measures and stochastic processes of random variables is the absence of a reference probability measure. This makes quite a bit of difference in what follows.

A process Y on a chain Γ of subalgebras of $\mathscr{B}$ is said to be a *supermartingale* provided that when $\mathscr{A}_1 \subset \mathscr{A}_2$ are in Γ then $Y_{\mathscr{A}_1} \geqslant E(Y_{\mathscr{A}_2}|\mathscr{A}_1)$. Y is said to be a *martingale* if $Y_{\mathscr{A}_1} = E(Y_{\mathscr{A}_2}|\mathscr{A}_1)$ in the same circumstance. Given a supermartingale Y one wishes to find a decomposition $Y = M - A$ where M is a martingale and A is an "increasing process" with $\inf\{\|A_{\mathscr{A}}\|\colon \mathscr{A} \in \Gamma\} = 0$. One definition of increasing process is given in Armstrong [**1983**]. If $\{\mu_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ is a family in $\mathrm{BA}(\mathscr{B})$ so that $\mu_{\mathscr{A}_1} \leqslant \mu_{\mathscr{A}_2}$ if $\mathscr{A}_1 \subset \mathscr{A}_2$ then this family is said to be *increasing* and $Y = \{E(\mu_{\mathscr{A}}|\mathscr{A})\colon \mathscr{A} \in \Gamma\}$ is called an *increasing process* on Γ. Decreasing processes are analogously defined. It is shown in §6 of Armstrong [**1983**] that every supermartingale Y does admit a decomposition $Y = M - A$ where M is a martingale and A is increasing with $\inf\{\|A_{\mathscr{A}}\|\colon \mathscr{A} \in \Gamma\} = 0$. In fact, Y is a decreasing process. This is used to show that the class of differences of nonnegative finitely additive supermartingales (i.e., F-processes) is the class of differences of nonnegative submartingales. One consequence is the existence of at least one P in $\mathrm{BA}_1^+(\mathscr{B})$ (the finitely additive probability measures on $\mathscr{B}$) so that $Y_{\mathscr{A}} \ll E(P|\mathscr{A})$ for all $\mathscr{A} \in \Gamma$. Here $\ll$ denotes the $\varepsilon - \delta$ notion of absolute continuity standard for finitely additive measures. Unfortunately, the increasing processes are not the precise analogue of increasing stochastic processes adapted to a filtration but are more general as well be seen.

One may convert finitely additive processes on a chain Γ of subalgebras of $\mathscr{B}$ into countably additive processes on a filtration of σ-algebras. This may be done by passage to the totally disconnected compact Hausdorff Stone space $X_{\mathscr{B}}$. The clopen algebra of $X_{\mathscr{B}}$ is isomorphic to $\mathscr{B}$ under a map $A \to [A]$ from $\mathscr{B}$ to the clopen algebra $[\mathscr{B}]$. If $\mathscr{A}$ is a subalgebra of $\mathscr{B}$ then $\mathscr{A}$ is isomorphic to $[\mathscr{A}] = \{[A]: A \in \mathscr{A}\} \subset \mathscr{B}$. Denote by $\mathscr{F}_{\mathscr{B}}$ the Baire algebra $\sigma([\mathscr{B}])$ of $X_{\mathscr{B}}$. For $\mathscr{A}$ a subalgebra of $\mathscr{B}$ denote by $\mathscr{F}_{\mathscr{A}}$ the σ-algebra $\sigma([\mathscr{A}])$. It may be shown that $\mathscr{F}_{\mathscr{A}}$ is isomorphic to the Baire algebra of the Stone space $X_{\mathscr{A}}$ of $\mathscr{A}$ upon identifying $[A]$ or A with the corresponding clopen set in $X_{\mathscr{A}}$. The Stone correspondence identifies a $\mu \in \mathrm{BA}(\mathscr{B})$ with a unique $\tilde{\mu} \in \mathscr{M}(X_{\mathscr{B}})$ (the Radon measures on $X_{\mathscr{B}}$) via the formula $\mu(A) = \tilde{\mu}([A])$ for all $\mathscr{A} \in \mathscr{B}$. The map $\mu \to \tilde{\mu}$ is a Banach lattice isomorphism from $\mathrm{BA}(\mathscr{B})$ onto $\mathscr{M}(X_{\mathscr{B}})$. $\mathscr{M}(X_B)$ is considered to be $\mathrm{CA}(\mathscr{F}_{\mathscr{B}})$, the countably additive Baire measures on $X_{\mathscr{B}}$, in the usual fashion. If μ is in $\mathrm{BA}(\mathscr{A})$ for some subalgebra $\mathscr{A}$ of $\mathscr{B}$ then $\tilde{\mu}$ is defined in the same manner as above as an element of $\mathrm{CA}(\mathscr{F}_{\mathscr{A}})$. It is easily seen that if $Y = \{Y_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ is a process on a chain Γ of subalgebras of $\mathscr{B}$ then $\tilde{Y} = \{\tilde{Y}_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ is a countably additive process on $\{\mathscr{F}_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ and that all countably additive processes on $\{\mathscr{F}_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ arise in this fashion. A process Y is a martingale, supermartingale or increasing process iff $\tilde{Y}$ is. Y is such that $Y_{\mathscr{A}} \ll E(P|\mathscr{A})$ for all $\mathscr{A}$ for some $P \in \mathrm{BA}_1^+(\mathscr{B})$ iff $\tilde{Y}_{\mathscr{A}} \ll E(\tilde{P}|\mathscr{F}_{\mathscr{A}})$ for all $\mathscr{A}$. In this case, let $f^P_{\mathscr{A}}$ be the density of $\tilde{Y}_{\mathscr{A}}$ with respect to $E(\tilde{P}|\mathscr{F}_{\mathscr{A}})$ for all $\mathscr{A}$. The family $f^P = \{f^P_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ is a stochastic process adapted to the filtration $\{\mathscr{F}_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ which is a supermartingale, martingale, increasing process iff Y is.

An increasing process $A = \{A_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ is said to be *increasing with respect to* $P \in \mathrm{BA}_1^+(\mathscr{B})$ iff it is of the form $\{E(\mu_{\mathscr{A}}|\mathscr{A}): \mathscr{A} \in \Gamma\}$ with $\{\mu_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ an increasing family of measures absolutely continuous to P. If $g^P_{\mathscr{A}}$ is the $\tilde{P}$-density of $\tilde{\mu}_{\mathscr{A}}$ we have $g^P = \{g^P_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ increasing in $L^1(X_{\mathscr{B}}, \mathscr{F}_{\mathscr{B}}, \tilde{P})$. It need not be the case that g^P is adapted to $\{\mathscr{F}_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$. If this is the case we say that A is an *adapted increasing process with respect to P*. An equivalent requirement is that $h^P = \{E(g^P_{\mathscr{A}}|\mathscr{F}_{\mathscr{A}}): \mathscr{A} \in \Gamma\}$ be increasing in $L^1(X_{\mathscr{B}}, \mathscr{F}_{\mathscr{B}}, \tilde{P})$, for $\{\mu_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ could be replaced by $\{\nu_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ where $\tilde{\nu}_{\mathscr{A}} = h^P_{\mathscr{A}}\tilde{P}$ for $\mathscr{A} \in \Gamma$. One may give a necessary and sufficient condition purely in terms of the finitely additive measures $\{\mu_{\mathscr{A}}: \mathscr{A} \in \Gamma\}$ and P so that A be adapted. This is that each $\mu_{\mathscr{A}}$ is the limit in variation norm of a sequence $\{\mu^n_{\mathscr{A}}: n \in N\}$ where each $\mu^n_{\mathscr{A}}$ is obtained from P by choosing a finite partition $\{A_{nj}: j = 1, \ldots, m\} \subset \mathscr{A}$ scalars $\{\lambda^{\mathscr{A}}_{nj}: j = 1, \ldots, m\}$ and letting $\mu^n_{\mathscr{A}}(B) = \sum_{j=1}^m \lambda^{\mathscr{A}}_{nj} P(B \cap A_{nj})$. This is an exercise in the Bochner finitely-additive Radon-Nikodym theorem (Bochner and Phillips [**1941**]). We are nearly in a position to show that every finitely additive supermartingale is the difference of a martingale and an adapted increasing process with respect to some $P \in \mathrm{BA}_1^+(\mathscr{B})$. The only obstacle is the extension of Doob-Meyer Decomposition Theorem from filtrations indexed by $[0, \infty)$ to filtrations indexed by arbitrary linearly ordered set. This is not a great obstacle, for a supermartingale has its variation concentrated on an order separable set and such sets are isomorphic to subsets of $[0, \infty)$.

PROPOSITION A. *Let* $\{\mathscr{F}_t: t \in T\}$ *be a linearly ordered filtration in a probability space* $(X, \mathscr{F}, P)$. *In order that an* L^1*-bounded supermartingale* $\{f_t: t \in T\}$ *adapted to* $\{\mathscr{F}_t: t \in T\}$ *admit a decomposition* $f = m - a$ *where* m *is a martingale and* a *is an increasing process with* $\inf\{\|a_t\|_1: t \in T\} = 0$ *it is necessary and sufficient that* $\{f_\tau: \tau$ *simple* T*-valued stopping time* $\leqslant t\}$ *be uniformly integrable for all* $t \in T$.

PROOF. The case where T is order separable is considered first. In this case T is order isomorphic to a subset of $[0, \infty)$ with $0 = \inf T$. Regard T as actually being a subset of $[0, \infty)$. Extend the filtration from T to $[0, \infty)$ as follows. If $t \in [0, \infty)$, set $\mathscr{F}_t = \bigcap\{\mathscr{F}_s\colon\ t \leqslant s, s \in T\}$. Extend f from T to $[0, \infty)$ by setting $f_t = \sup\{E(f_s|\mathscr{F}_t)\colon\ t \leqslant s \in T\}$. We have $\{f_\tau\colon\ \tau$ simple stopping time $\leqslant t\}$ uniformly integrable for $0 \leqslant t < \infty$ iff $\{f_\tau\colon\ \tau$ simple T-valued stopping time $\leqslant t\}$ is uniformly integrable for $t \in T$. The reason for this is that uniform integrability of a family is preserved by adjoining to it arbitrary conditional expectations on sub-σ-algebras or sequential almost sure limits. Thus, if $\{f_\tau\colon\ \tau$ simple T-valued stopping time $\leqslant t\}$ is uniformly integrable for $t \in T$ then $f_t = m_t - a_t$ for $0 \leqslant t \leqslant \infty$ with m a martingale and a an increasing process with $a_0 = 0$. Restriction of f, m, and a to T gives us the desired decomposition. Conversely, if $f_t = m_t - a_t$ for $t \in T$ with m a martingale and a an increasing process with $\inf_T a_t = 0$ then $\{m_\tau\colon\ \tau$ simple stopping time $\leqslant t\}$ is uniformly integrable for $t \in T$ and, since $\{a_\tau\colon\ \tau$ simple stopping time $\leqslant t\}$ is dominated by a_t, it is uniformly integrable for $t \in T$. As a result $\{f_\tau = m_\tau - a_\tau\colon\ \tau$ simple stopping time $\leqslant t\}$ is uniformly integrable for $t \in T$. This establishes the order separable case.

In order to establish the proposition for nonseparable T it is only necessary to show that if $\{f_\tau\colon\ \tau$ simple T-valued stopping time $\leqslant t\}$ is uniformly integrable for all $t \in T$ then f admits a Doob-Meyer decomposition. The converse is established as in the preceding paragraph. Let $R \subset [0, \infty)$ denote $\{\|f_t\|_1\colon\ t \in T\}$. If $r \in \overline{R}\setminus R$ and r is a limit from above of $\{\|f_t\|_1\colon\ \|f_t\|_1 > r\}$, set $\mathscr{F}_{r^-}$ equal to $\sigma\{\mathscr{F}_t\colon\ \|f_t\|_1 > r\}$. If $r \in \overline{R}\setminus R$ is a limit from below of $\{\|f_t\|\colon\ \|f_t\| < r\}$, set $\mathscr{F}_{r^+}$ equal to $\bigcap\{\mathscr{F}_t\colon\ \|f_t\| < r\}$. If $r \in R$ let $L_r = \{t \in T\colon\ \|g_t\| = r\}$. In this case set $\mathscr{F}_{r^+}$ equal to $\sigma\{\mathscr{F}_t\colon\ \|g_t\| = r\}$ and $\mathscr{F}_{r^-}$ equal to $\bigcap\{\mathscr{F}_t\colon\ \|g_t\| = r\}$. Adjoin $\{r^-$ and/or $r^+\colon\ r \in \overline{R}\}$ to T to obtain $T^\sharp$ where $s \leqslant r^- \leqslant r^+ \leqslant t$ if $\|g_s\| > r > \|g_t\|$ and $r^- < t < r^+$ if $\|g_t\| = r$. The filtration $\{\mathscr{F}_t\colon\ t \in T^\sharp\}$, as defined, extends $\{\mathscr{F}_t\colon\ t \in T\}$. Since $\{f_s\colon\ s \leqslant t\}$ is uniformly integrable for $t \in T$ we may extend f to be a supermartingale adapted to the filtration $\{\mathscr{F}_t\colon\ t \in T^\sharp\}$ by setting $f_{r^-} = \lim\{f_t\colon\ \|f_t\| > r\}$ or

$$f_{r^+} = \lim\{f_t\colon\ \|f_t\| < r\}$$

if $r \in \overline{R}\setminus R$ and by setting $f_{r^-} = \lim\{f_t\colon\ t$ decreasing in $L_r\}$ and

$$f_{r^+} = \lim\{f_t\colon\ t \text{ increasing in } L_r\}$$

if $r \in R$. It is easily verified that $\{f_\tau\colon\ \tau$ simple $T^\sharp$-valued stopping time $\leqslant t\}$ is uniformly integrable for each $t \in T^\sharp$. The ensemble $\tilde{T}$ of all r^- or r^+ for $r \in \overline{R}$ is obtained by deletion of L_r from $T^\sharp$ for all $r \in R$. One may verify that if D is a countable dense set in R then $\{r^+, r^-\colon\ r \in D\}$ is a countable order dense set in $\tilde{T}$. One has $\{f_\tau\colon\ \tau$ simple $\tilde{T}$-valued stopping time $\leqslant t\}$ uniformly integrable for all $t \in \tilde{T}$. As a result, it may be concluded that, for all $t \in \tilde{T}$, $f_t = m_t - a_t$ where m is a martingale and a is an increasing process with $0 = \inf\{a_t\colon\ t \in \tilde{T}\}$. Notice that if $r \in R$ then $a_{r^-} = a_{r^+}$ almost surely for $f_{r^-} = f_{r^+}$ almost surely. As a result, a may be extended to $T^\sharp$ from $\tilde{T}$ as an increasing process. If $t \in L_r$ then a_t must be the function a_{r^-}. Extend m as a martingale from $\tilde{T}$ to $T^\sharp$. We have $f = m - a$ on $T^\sharp$. Upon restriction from $T^\sharp$ to T we obtain the desired Doob-Meyer decomposition of f. This completes the proof of the proposition. $\square$

THEOREM B. *Let Y be a nonnegative bounded finitely additive supermartingale on a chain Γ of subalgebras of a Boolean algebra $\mathscr{B}$. There is a $P \in \mathrm{BA}_1^+(\mathscr{B})$ and a $Q \in \mathrm{BA}^+(\mathscr{B})$ with $Q \ll P$ so that if $\mathscr{M}Q$ is the martingale $\{E(Q|\mathscr{A})\colon \mathscr{A} \in \Gamma\}$ then $A = \mathscr{M}Q - Y$ is an adapted increasing process with respect to P.*

PROOF. By our previous remarks we know that $Y = M - A$ where A is an increasing process with $\inf\{\|A_{\mathscr{A}}\|\colon \mathscr{A} \in \Gamma\} = 0$ and M is a martingale. We may find $\nu \in \mathrm{BA}^+(\mathscr{B})$ so that $E(\nu|\mathscr{A}) = M_{\mathscr{A}}$ for $\mathscr{A} \in \Gamma$. Since $E(\nu|\mathscr{A}) = M_{\mathscr{A}}$ it follows that $Y_{\mathscr{A}} \leqslant E(\nu|\mathscr{A})$ for $\mathscr{A} \in \Gamma$. Let P be $\lambda\nu$ where $\lambda = \|\nu\|^{-1}$. It follows that $Y_{\mathscr{A}} \leqslant \lambda E(P|\mathscr{A})$ for $\mathscr{A} \in \Gamma$. Let us pass to the Stone space $X_{\mathscr{B}}$ equipped with the filtration $\{\mathscr{F}_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ as in previous remarks. Let $\tilde{P} \in \mathscr{M}_1^+(\mathscr{B})$ correspond to P under the Stone correspondence. Let $\tilde{Y} = \{\tilde{Y}_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ be the countably additive supermartingale on $\{\mathscr{F}_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ corresponding to Y under the Stone correspondence. Let $\{f_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\} = f$ be the supermartingale adapted to $\{\mathscr{F}_{\mathscr{A}}\colon \mathscr{A} \in \Gamma\}$ with $\tilde{Y}_{\mathscr{A}} = f_{\mathscr{A}} E(\tilde{P}|\mathscr{F}_{\mathscr{A}})$ for $\mathscr{A} \in \Gamma$. The supermartingale f is uniformly bounded in L^∞-norm by λ hence is uniformly integrable. Proposition A may be applied to f to yield a decomposition $f = m - a$ where m is a martingale and a is an increasing process with $\inf\{\|a_{\mathscr{A}}\|_1\colon \mathscr{A} \in \Gamma\} = 0$. Since m is uniformly integrable it is of the form $\{E(h|\mathscr{F}_{\mathscr{A}})\colon \mathscr{A} \in \Gamma\}$ for some $h \in L^{1+}(X_{\mathscr{B}}, \mathscr{F}_{\mathscr{B}}, \tilde{P})$. Let $\tilde{Q} = h\tilde{P}$ and let $Q \in \mathrm{BA}^+(\mathscr{B})$ correspond to $\tilde{Q}$ under the Stone correspondence. If $\mathscr{A} \in \Gamma$ then $A_{\mathscr{A}} = E(Q|\mathscr{A}) - Y_{\mathscr{A}}$ corresponds to $a_{\mathscr{A}}\tilde{P}$ under the Stone correspondence. It is immediate that this Q is the one which we were seeking. □

We may obtain the following as a nearly immediate corollary relating to ordinary supermartingales adapted to a filtration:

COROLLARY C. *Let $(X, \mathscr{F}, P)$ be a probability measure space, $\{\mathscr{F}_t\colon t \in T\}$ be a linearly ordered filtration of sub-σ-algebras of $\mathscr{F}$ and $\{f_t\colon t \in T\}$ be a nonnegative L^1-bounded supermartingale adapted to $\{\mathscr{F}_t\colon t \in T\}$. There is a $\Phi \in L_1^{\infty*+}(X, \mathscr{F}, P)$ and a $\Psi \in L^{\infty*+}(X, \mathscr{F}, P)$ with $\Psi \ll \Phi$ so that $A = \{A_t\colon t \in T\} \subset L^{\infty*+}(X, \mathscr{F}, P)$ defined by $A_t = E(\Psi|\mathscr{F}_t) - f_t P$ is an adapted increasing process with respect to Φ.*

PROOF. $L^{\infty*}(X, \mathscr{F}, P)$ may be identified with $\mathrm{BA}(\mathscr{F}_P)$ where $\mathscr{F}_P$ is the measure algebra of P obtained as the Boolean quotient of $\mathscr{F}$ by the ideal of P-negligible sets. The filtration $\{\mathscr{F}_t\colon t \in T\}$ induces, under the quotient map, a chain $\{(\mathscr{F}_t)_P\colon t \in T\}$ of subalgebras of $\mathscr{F}_P$. $\{f_t P\colon t \in T\}$ may be considered as a finitely additive supermartingale on this chain. Theorem B applies in this situation to establish the corollary. □

REMARKS. (a) Corollary C states that a Doob-Meyer decomposition exists for any supermartingale if we are allowed to change the reference probability measure from P to an element Φ of $\mathrm{BA}_1^+(\mathscr{F}_P)$. Note that Φ is weakly absolutely continuous with respect to P in annihilating all P-negligible sets. If Φ were to be countably additive it would be absolutely continuous with respect to P hence of the form hP for some $h \in L_1^{1+}(X, \mathscr{F}, P)$. In this case, Ψ is of the form ghP for some g in $L^{1+}(X, \mathscr{F}, hP)$ and A_t is of the form $a_t hP$ where $\{a_t\colon t \in T\}$ is an increasing process. In this case, it follows that $\{ f_\tau$: τ-simple T-valued stopping time $\leqslant t \}$ is uniformly integrable for $t \in T$. Thus, if $P = \Phi$ is an impossible choice for Φ then any suitable Φ must have a nontrivial purely finitely additive part. The work of Astbury [**1981**] is related to this.

(b) Another connection between finite additivity and the Doob-Meyer Decomposition Theorem is the characterization in Metivier and Pellaumail [**1975**] of those nonnegative supermartingales admitting such a decomposition as those whose Doleans-Föllmer measure on the ring of predictable sets is countably additive. Thus, those which do not admit such a decomposition have a nontrivial purely finitely additive part in their Doleans-Föllmer measure. The relationship between this observation and the remarks in (a) has yet to be made precise.

(c) It need not be the case that the Doob-Meyer decomposition for countably additive supermartingales leads one inevitably to finitely additive measures. Consider $\{ X_t$: $t \in T \}$ a linearly ordered projective system of compact Hausdorff spaces with $P_{t,s}$: $X_s \to X_t$ the associated continuous surjection for $s \geqslant t$. Let X_∞ be the projective limit with P_t: $X_\infty \to X_t$ the associated projection for $t \in T$. Let $\mathscr{F}$ be the Baire field of X_∞ and for $t \in T$ let $\mathscr{B}_t$ be the Baire field of X_t. Let $\mathscr{F}_t = P_t^{-1}(\mathscr{B}_t) \subset \mathscr{F}$ for all $t \in T$. One may consider countably additive processes adapted to the filtration $\{ \mathscr{F}_t$: $t \in T \}$. Equip $\mathrm{CA}^+(\mathscr{B}_t)$, hence $\mathrm{CA}^+(\mathscr{F}_t)$, with the vague topology so closed norm bounded sets are compact. The canonical surjection $\mathscr{A}_{t,s}$: $\mathrm{CA}^+(\mathscr{F}_s) \to \mathrm{CA}^+(\mathscr{F}_t)$ associated with $P_{t,s}$ is vaguely continuous. The proof of Theorem 6-1 of Armstrong [**1983**] is valid in this setting as is that of Theorem B. As a result, if Y is a countably additive norm bounded nonnegative supermartingale adapted to $\{ \mathscr{F}_t$: $t \in T \}$ there is a $P \in \mathrm{CA}_1^+(\mathscr{F}_\infty) = \mathscr{M}_1^+(X_\infty)$, and a $Q \in \mathrm{CA}^+(\mathscr{F}_\infty)$ with $Q \ll P$ so that, if $M_t \in \mathrm{CA}^+(\mathscr{F}_t)$ is the restriction of Q to $\mathscr{F}_t$ for all t, then $A_t = M_t - A_t$ for $t \in T$ yields an adapted increasing process with respect to P with

$$\inf\left\{ \| A_t \| \colon t \in T \right\} = 0.$$

(d) As we have remarked previously we have not considered uniqueness in the Doob-Meyer Decomposition Theorem. This is of considerable importance for the purpose of stochastic integration. If one considers supermartingales adapted to a linearly ordered filtration of sub-σ-algebras of a probability measure space it appears clear that the proof of Proposition A yields a predictable increasing process in the Doob-Meyer decomposition from a predictable increasing process on the order separable $\tilde{T}$ and conversely. Uniqueness of the predictable process on T appears to follow from uniqueness on $\tilde{T}$. As a result it appears that, with the appropriate definition of predictability of finitely additive processes, there is a unique predictable adapted increasing process with respect to P which occurs in a Doob-Meyer decomposition of a finitely additive supermartingale in Theorem B. Of course, different P in Theorem A give rise to different predictable adapted increasing processes.

(e) The effects of change in the reference probability measures on semimartingales have been considered in several places. Memin [**1980**] is a notable example. Usually the new probability is one mutually absolutely continuous with respect to the original and in these cases always countably additive.

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