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ELLIPTIC OPERATORS AND A THEOREM OF POINCARÉ

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ABSTRACT. A general vanishing theorem is proved for elliptic operators. This result is then used to give a simple proof of the fact that the arithmetic genus vanishes for complex manifolds of odd dimension n with nowhere zero $(n, 0)$ form.

1. Introduction and statement of main result. Let M^n be a closed complex manifold of complex dimension n. Recall that the Hodge numbers $h^{p,q}$, where $0 \leqslant p, q \leqslant n$, are the dimensions of the sheaf cohomology groups $H^q(M, \Omega^p)$ and that one defines $\chi^p = \sum_{q=0}^n (-1)^q h^{p,q}$. It is known that the integer χ^p is a Chern number since it is the index of a certain elliptic operator [2]. The integer χ^0 is called the arithmetic genus M^n, and it turns out that the Euler characteristic $\chi(M) = \Sigma(-1)^p \chi^p$.

According to a classical result of Poincaré, if an m-manifold admits a nonsingular vector field then its Euler characteristic $\chi(M^m) = 0$. If $\dim_R M = m = 2$ then we can view M as a Riemann surface and the Euler characteristic is twice the arithmetic genus, since $\chi = \chi^0 - \chi^1$ and $\chi^1 = -\chi^0$ (e.g., since $h^{0,1} = h^{1,0}$ for every Riemann surface). Also, the real vector field X is nonsingular if and only if the $(1, 0)$ vector field $Z = X - iJX$ is nonsingular, where J is the complex structure. Thus, Poincaré's result can be reformulated as: If a Riemann surface admits a nonsingular $(1, 0)$ vector field (or $(1, 0)$ form) then its arithmetic genus vanishes. We generalize this result as follows:

THEOREM A. *If a closed complex manifold of odd (complex) dimension n admits a nowhere vanishing smooth form of type* $(n, 0)$ *then its arithmetic genus is zero.*

The proof of the theorem is given in §3 and is based on a general index vanishing theorem for elliptic operators (to be proved in §2). These results are contained in a section of the author's doctoral dissertation [5].

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2. A general index vanishing theorem. We take [6 and 7] as general references on elliptic operators and Fredholm theory.

Let E and F be hermitian (but not necessarily holomorphic) vector bundles over a closed manifold M with volume form d vol. We thus have fiber metric $\langle \theta, \psi \rangle_E(x)$

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and global metric $(\theta, \psi)_E = \int \langle \theta, \psi \rangle_E(x)\, d\operatorname{vol}(x)$ for E and, similarly, for F. If P: $\Gamma(E) \to \Gamma(F)$ is an elliptic operator of order m then P is Fredholm and its index $\operatorname{ind}(P) =_{\text{def}} \dim\ker P - \dim\ker P^*$, where P^* is the formal adjoint of P.

THEOREM B. *Let E, F, and P be as above with order $P = 1$. If there exists a bundle map Q: $\Gamma(E) \to \Gamma(F)$ such that*

(i) *$P^*Q + Q^*P$ is a differential operator of order 0 and*

(ii) *Q^*Q is positive definite, i.e., $(Q^*Q\phi, \phi)_E \geqslant c(\phi, \phi)_E$ for some $c > 0$, $\forall \phi \in \Gamma(E)$,*

then $\operatorname{ind}(P) \leqslant 0$. *If the same relations also exist between P^* and Q^* then* $\operatorname{ind}(P) = 0$.

PROOF. Let $P_t =_{\text{det}} P + tQ$, a continuous path of Fredholm operators connected to P. It follows that $\operatorname{ind} P = \operatorname{ind}(P_t)$, $\forall t \geqslant 0$. We will show that $\operatorname{ind}(P_t) \leqslant 0$ for all $t \geqslant$ some t_0.

To this end, we consider $P_t^*P_t$ since $\ker(P_t^*P_t) = \ker P_t$. Now $P_t^*P_t = P^*P + tR + t^2Q^*Q$ where $R = Q^*P + P^*Q$ is a bounded operator. Set $A(t) = P^*P + t^2Q^*Q$. Then

$$\|A(t)\phi\|_E\|\phi\|_E \geqslant (A(t)\phi, \phi)_E = \|P\phi\|_E^2 + t^2(Q^*Q\phi, \phi)_E \geqslant t^2c\|\phi\|_E^2.$$

Thus $\ker A(t) = 0$. Since $A(t)$ has closed range and $\ker A^*(t) = \ker A(t) = 0$, $\overline{A(t)}$ exists and $\|A^{-1}(t)\| \leqslant \alpha t^{-2}$, $\alpha = 1/c$. It follows that for all $t \geqslant$ some t_0 the operator $C(t) = A(t)^{-1} \circ tR$ is bounded with norm < 1. However, if $t \geqslant t_0$, and $\phi \in \ker P_t^*P_t$ then

$$0 = P_t^*P_t\phi = [A(t) + tR]\phi$$

and so $(I + C(t))\phi = 0$. It follows that $\phi = 0$ and so $\ker(P_t) = 0$ as required. A repetition of the argument, with P and Q replaced with their adjoints, completes the proof.

3. The proof of Theorem A. Let $A^{r,s}$ denote the space of smooth sections of the complex bundle $\Lambda^{r,s}$ of forms of bidegree (r, s), and set $\Lambda^{r,\text{even}} = \bigoplus_{s \geqslant 0} \Lambda^{r,2s}$ and $\Lambda^{r,\text{odd}} = \bigoplus_{s \geqslant 0} \Lambda^{r,2s+1}$. Now let

$$E = \Lambda^{0,\text{even}} \oplus \Lambda^{n,\text{odd}}, \qquad F = \Lambda^{0,\text{odd}} \oplus \Lambda^{n,\text{even}}.$$

Choose an hermitian metric and let the adjoint of $\bar\partial$ be denoted $\bar\partial^*$. Then $D_q = \operatorname{def} \bar\partial + \bar\partial^*$: $A^{q,\text{even}} \to A^{q,\text{odd}}$ and $D_q^* = (\bar\partial + \bar\partial^*)^* = \bar\partial + \bar\partial^*$: $A^{q,\text{odd}} \to A^{q,\text{even}}$. Define

$$P = \begin{bmatrix} D_0 & 0 \\ 0 & D_n^* \end{bmatrix} : \Gamma(E) \to \Gamma(F).$$

As is well known (and easy to see) the operators D_0 and D_n are elliptic with indices χ^0 and χ^n, respectively. By Serre Duality (see [3]) $\chi^p = (-1)^n\chi^{n-p}$ and so $\operatorname{ind}(P) = \operatorname{ind}(D_0) + \operatorname{ind}(D_n^*) = \chi^0 - \chi^n = 2\chi^0$. We will show that $\operatorname{ind}(P) = 0$. To this end, we let ω be the nowhere zero $(n,0)$ form and denote by L_ω the operation of left exterior multiplication by ω. Thus

$$L_\omega\colon A^{0,\mathrm{even}} \to A^{n,\mathrm{even}}, \qquad L_\omega\colon A^{0,\mathrm{odd}} \to A^{n,\mathrm{odd}},$$

and $L_\omega A^{n,*} = 0$. Also, set $L_\omega^* =$ the hermitian adjoint of L_ω, so that $L_\omega^* A^{0,*} = 0$, and let

$$Q = \begin{bmatrix} 0 & L_\omega^* \\ L_\omega & 0 \end{bmatrix} \colon \Gamma(E) \to \Gamma(F),$$

and note that Q "twists" the summands. The desired relation $\operatorname{ind}(P) = 0$ now follows from Theorem B together with

LEMMA 1. *$P^*Q + Q^*P$ is a differential operator of order zero.*

LEMMA 2. *$Q^*Q = QQ^* = Q^2$ is positive definite.*

To prove Lemma 1 we note that as matrices of operators (without domains) $P^* = P$ and $Q^* = Q$ so that

$$P^*Q + Q^*P = PQ + QP = \begin{bmatrix} 0 & DL_\omega^* + L_\omega^* D \\ DL + LD & 0 \end{bmatrix} = \begin{bmatrix} 0 & S + R^* \\ R + S^* & 0 \end{bmatrix}$$

where $R = \bar\partial L_\omega + L_\omega \bar\partial =$ left multiplication by $\bar\partial\omega \in \Lambda^{n,1}$, and $S = L_\omega^*\bar\partial + \bar\partial L_\omega^*$. Thus R is clearly of order zero. To show that S is of order zero we note that, locally, $\omega = \zeta_1 \wedge \cdots \wedge \zeta_n$ for some orthogonal $(1,0)$ forms $\{\zeta_1, \ldots, \zeta_n\}$. Thus $L_\omega^* = C_n \circ \cdots \circ C_1$, where C_j denotes contraction with the vector field of type $(1,0)$ that is dual to ζ_j. It is easy to check that

$$L = \bar\partial C_z + C_z \bar\partial$$

is an operator of order zero if Z is a vector field of type $(1,0)$. Indeed, the symbol of L, $\sigma(\xi_x)$, is a constant multiple of $C_z(\bar\partial f) = 0$ if $df = \xi_x$ and $f(x) = 0$. It follows that $\bar\partial L_\omega^* = \bar\partial(C_n \circ \cdots \circ C_1) = (L_n - C_n\bar\partial)C_{n-1} \circ \cdots \circ C =$ operator of order zero $- C_n\bar\partial C_{n-1} \circ \cdots \circ C_1$. Continuing in this fashion, and recalling that n is odd, $\bar\partial L^* =$ operator of order zero $- (C_n \circ \cdots \circ C_1)\bar\partial$, as claimed.

To show that Q^*Q is positive definite we again write, locally, $\omega = \zeta_1 \wedge \cdots \wedge \zeta_n$ and $L_\omega^* = C_n \circ \cdots \circ C_1$. It follows that $L_\omega^* L_\omega \alpha = \prod \|\zeta_j\|^2 \alpha = \|\omega\|^2 \alpha$ if $\alpha \in \Lambda^{0,*}$ and $L_\omega L_\omega^* \beta = \|\omega\|^2 \beta$, $\beta \in \Lambda^{n,*}$. Since $\|\omega\|^2 \geqslant$ constant > 0 the proof of Lemma 2, and with it the proof of the theorem, is complete.

REMARKS. (a) Essentially the same method of proof yields the standard result that the existence of a nonsingular vector field (or 1-form ζ) on M^n forces the index of $d + d^*\colon A^{\mathrm{even}} \to A^{\mathrm{odd}}$, i.e. the Euler characteristic, to vanish:

$$\operatorname{ind}(d + d^*) = \operatorname{ind}\big(d + d^* + t\big(L_\zeta + L_\zeta^*\big)\big) = 0$$

for t sufficiently large, since $(L_\zeta + L_\zeta^*)^*(L_\zeta + L_\zeta^*)$ is positive definite.

(b) A variant of the method of this paper gives an elliptic proof of a classical theorem of Hurwitz [**4**]: *If M^n is a closed 2-manifold with $\chi(M) < 0$, then the isometry group of M^n is finite for any metric* [or in its original form (which is equivalent via uniformization): A Riemann surface of genus > 1 admits only a finite number of conformal automorphisms]. For the proof, we let g be any metric and show that (M, g) admits no infinitesimal isometries, i.e., that the Lie algebra of the group of isometries is trivial, and consequently, since the group G of isometries is compact, G is finite.

(c) If there did exist a nontrivial infinitesimal isometry X with dual form $\xi = g(x, \cdot)$, then consider the operator $D = d + d^* + L_\xi + L_\xi^*\colon A^{\text{even}} \to A^{\text{odd}}$, where $L_\xi \alpha = \xi \wedge \alpha$. Since $L_\xi + L_\xi^*$ is of order zero, we have $\chi(M^2) = \operatorname{ind}(d + d^*) = \operatorname{ind}(D)$. Now $\ker(D^*) = \ker(DD^*)$. It is easy to check that $DD^* = \Delta + |\xi|^2 + \theta_x + \theta_X^*$ on $\operatorname{Dom}(D^*) =$ 1-forms. Here θ_x denotes Lie derivative with respect to X. Now $\theta_x + \theta_x^*$ is always an operator of order zero, and when X is an infinitesimal isometry $\theta_x + \theta_x^* \equiv 0$ (a short proof, courtesy of I. M. Singer: $\theta_x^* = -*\theta_x*$ so that $\theta_x + \theta_x^* = -*[\theta_x, *] = 0$ because the Hodge star operator commutes with isometries). It follows that if $\alpha \in \ker DD^*$ then $\Delta\alpha + |\xi|^2\alpha = 0$. Multiplying by α and integrating by parts, we find that the 1-form $\alpha \equiv 0$ on the set where $|\xi|^2 > 0$. It follows from the unique continuation theorem [**1**] that $\alpha \equiv 0$ on M and $\ker D^* = \{0\}$. That is, $\chi(M) = \dim\ker D \geqslant 0$.

(d) The techniques of this paper generalize easily to treat operators on forms with values in holomorphic vector bundles. We leave the details to the reader.

(e) The methods used here are closely related to Witten's techniques in [**8**]. This will be pursued elsewhere.

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