METRIC SPACES ON WHICH CONTINUOUS FUNCTIONS ARE UNIFORMLY CONTINUOUS AND HAUSDORFF DISTANCE

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ABSTRACT. Atsugi has internally characterized those metric spaces X for which each real-valued continuous function on X is uniformly continuous as follows: (1) the set X' of limit points of X is compact, and (2) for each $\varepsilon > 0$, the set of points in X whose distance from X' exceeds ε is uniformly discrete. We obtain these new characterizations: (a) for each metric space Y , the Hausdorff metric on $C(X, Y)$, induced by a metric on $X \times Y$ compatible with the product uniformity, yields the topology of uniform convergence; (b) there exists a metric space Y containing an arc for which the Hausdorff metric on $C(X, Y)$ yields the topology of uniform convergence; (c) the Hausdorff metric topology on $CL(X)$ is at least as strong as the Vietoris topology. We also characterize those metric spaces whose hyperspace is such a space.

Let $\langle W, d \rangle$ be a metric space. If $K \subset W$ and ε is positive, let $S_\varepsilon[K]$ denote the union of all open ε -balls whose centers run over K . If K_1 and K_2 are nonempty subsets of W , and for some $\varepsilon > 0$ both $S_\varepsilon[K_1] \supset K_2$ and $S_\varepsilon[K_2] \supset K_1$, then the *Hausdorff distance* h_d between them is given by the formula $h_d(K_1, K_2) = \inf\{\varepsilon: S_\varepsilon[K_1] \supset K_2 \text{ and } S_\varepsilon[K_2] \supset K_1\}$. Otherwise, we write $h_d(K_1, K_2) = \infty$. If we restrict h_d to the closed nonempty subsets $CL(W)$ of W , then h_d defines an infinite-valued metric. Basic facts about this metric can be found in Aubin [2], Castaing and Valadier [3], or Engelking [4]. Now let $C(X, Y)$ denote the continuous functions from a metric space $\langle X, d_X \rangle$ to a metric space $\langle Y, d_Y \rangle$. We denote *uniform distance* in $C(X, Y)$ by d_1 , that is,

$$d_1(f, g) = \sup\{d_Y(f(x), g(x)): x \in X\}.$$

Alternatively, we can identify members of $C(X, Y)$ with their graphs in $X \times Y$, and define the distance between functions to be the Hausdorff distance between their graphs, as induced by some metric compatible with the product uniformity. For definiteness and computational simplicity, we choose the *box metric*:

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

In the sequel, whenever f and g are continuous, we shall write $d_2(f, g)$ for $h_\rho(f, g)$.

It is easy to check that $d_2(f, g) \leq d_1(f, g)$, so that the topology of uniform convergence is always at least as strong as the one defined by d_2 . Furthermore, if X is compact and Y is arbitrary, then d_1 and d_2 are equivalent metrics on $C(X, Y)$ [7]. This equivalence has been the basis for numerous papers in approximation theory by Bl. Sendov, V. Popov, B. Penkov, V. Veselinov and their associates in Sofia (see, e.g., [9, 10 and 12]). However, d_1 and d_2 can be equivalent somewhat

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more generally. It is the main purpose of this article to characterize those spaces X for which d_1 and d_2 define the same topologies on $C(X, Y)$, independent of the choice of target space Y . We shall show that the desired spaces X are those for which

- (1) the set X' of limit points of X is compact,
- (2) for each $\varepsilon > 0$, the set $(S_\varepsilon[X'])^c$ is uniformly discrete, i.e., there exists $\delta > 0$ such that whenever x and y are distinct points of $(S_\varepsilon[X'])^c$, then $d_X(x, y) > \delta$.

Atsugi [1] has characterized such metric spaces in terms of their uniform properties in several ways, the most important of which was recently rediscovered by Hueber [5]: each real-valued continuous function on X is uniformly continuous. In the sequel we shall call those metric spaces that exhibit properties (1) and (2) *Atsugi spaces*. Not surprisingly, these spaces are exactly those for which each open cover has a Lebesgue number. Although this is not hard to prove directly, it also follows from basic results regarding the fine uniformity on a paracompact space (see, e.g., §36 of [15]). Each Atsugi space is complete; in fact, Toader [13] has essentially observed that X is an Atsugi space if and only if each sequence in X with distinct terms such that pairs of terms are arbitrarily close frequently has a cluster point. However, Atsugi spaces need not be locally compact (but see [14]). For example, let $\{e_n : n \in \mathbb{Z}^+\}$ be the usual orthonormal basis in l_2 , and let X denote the subspace of l_2 consisting of the origin plus all points of the form $(1/j)e_n$ with $\{n, j\} \subset \mathbb{Z}^+$. We also mention that the problem of describing those metrizable topological spaces which admit an Atsugi metric has been considered by Nagata [8], Levshenko [6], and Rainwater [11].

We find it convenient to use a sequential characterization of Atsugi spaces slightly different from those given in [1, 5, and 13].

DEFINITION. A sequence $\{x_n\}$ of distinct terms in a metric space $\langle X, d \rangle$ is called a *sequence of paired isolated points* if for each n , the point x_n is isolated in X , and $\lim_{n \rightarrow \infty} d(x_{2n-1}, x_{2n}) = 0$.

The straightforward proof of the next lemma is omitted.

LEMMA. A metric space $\langle X, d \rangle$ is an Atsugi space if and only if each sequence of paired isolated points in X has a cluster point, and each sequence in X' has a cluster point.

To show that the equivalence of d_1 and d_2 on $C(X, Y)$ forces X to be an Atsugi space under most circumstances, we need a preliminary lemma. Recall that a subset A of a set B in a metric space is called δ -dense in B if $S_\delta[A] \supset B$.

LEMMA. Let $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ be metric spaces. Let $x \in X'$, and let $\phi : [0, 1] \rightarrow Y$ be a path such that $\phi(0) \neq \phi(1)$. Then for each pair of positive numbers ε and δ there exist $\{f, g\} \subset C(X, Y)$ such that

- (i) both $g(S_\varepsilon[x])$ and $f(S_\varepsilon[x])$ are δ -dense in $\phi([0, 1])$;
- (ii) $g(z) = f(z) = \phi(0)$ whenever $d_X(z, x) \geq \varepsilon$;
- (iii) for every z , either $f(z) = \phi(0)$ or $g(z) = \phi(0)$.

PROOF. By the uniform continuity of ϕ there exists $n \in \mathbb{Z}^+$ such that $\{\phi(j/n) : 1 \leq j \leq n\}$ is δ -dense in $\phi([0, 1])$. Since x is a limit point of X , we can find $2n$ distinct points $\{x_1, \dots, x_n, w_1, \dots, w_n\}$ in $S_\varepsilon[x]$. Let $\{S_{\lambda_j}[x_j] : j = 1, \dots, n\} \cup \{S_{\delta_j}[w_j] : j = 1, \dots, n\}$ be disjoint balls, each lying within $S_\varepsilon[x]$. The desired

functions f and g may be defined by

$$f(z) = \begin{cases} \phi \left[\frac{j}{n} \cdot \left(1 - \frac{1}{\lambda_j} d_X(z, x_j) \right) \right] & \text{if } d_X(z, x_j) < \lambda_j, \\ \phi(0) & \text{otherwise;} \end{cases}$$

$$g(z) = \begin{cases} \phi \left[\frac{j}{n} \cdot \left(1 - \frac{1}{\theta_j} d_X(z, w_j) \right) \right] & \text{if } d_X(z, w_j) < \theta_j, \\ \phi(0) & \text{otherwise.} \end{cases}$$

THEOREM 1. *Let $\langle X, d_X \rangle$ be a metric space. The following are equivalent.*

- (a) X is an Atsugi space;
- (b) For each metric space Y the Hausdorff metric on $C(X, Y)$ induced by the box metric on $X \times Y$ yields the topology of uniform convergence;
- (c) There exists a metric space Y containing an arc for which the Hausdorff metric on $C(X, Y)$ induced by the box metric on $X \times Y$ yields the topology of uniform convergence.

PROOF. (a)→(b) By Theorem 1 of [1], each element of $C(X, Y)$ is uniformly continuous. Let $\{f_n\}$ be a sequence in $C(X, Y)$ d_2 -convergent to a continuous function f , and let ε be positive. Choose $\delta > 0$ such that whenever $d_X(x, z) < \delta$, then $d_Y(f(x), f(z)) < \varepsilon/2$. Let $\lambda = \min\{\varepsilon/2, \delta\}$, and choose N so large that $d_2(f_n, f) < \lambda$ for all $n > N$. Fix x and $n > N$, and pick z satisfying $\rho[(x, f_n(x)), (z, f(z))] < \lambda$. In particular, $d_X(x, z) < \lambda \leq \delta$, and we obtain

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f(z)) + d_Y(f(z), f(x)) < \lambda + \varepsilon/2 \leq \varepsilon.$$

It follows that $d_1(f_n, f) \leq \varepsilon$. This shows that d_2 is at least as strong as d_1 . As noted earlier, we always have $d_2 \leq d_1$.

(b)→(c) Trivial.

(c)→(a) Suppose that d_1 and d_2 define the same topology. We show that X satisfies our sequential characterization of an Atsugi space. First, suppose that $\{x_n\}$ is a sequence of paired isolated points with no cluster point. Let a and b be distinct points of Y , and define $f: X \rightarrow Y$ by

$$f(z) = \begin{cases} a & \text{if } z = x_{2n} \text{ for some } n, \\ b & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{Z}^+$ let f_n agree with f except at the points x_{2n-1} and x_{2n} , where the values of f are reversed; that is,

$$f_n(z) = \begin{cases} a & \text{if } z = x_{2n-1}, \\ b & \text{if } z = x_{2n}, \\ f(z) & \text{otherwise.} \end{cases}$$

Since $\{x_n\}$ has no cluster point, f and f_n are continuous. For each n , $d_1(f_n, f) = d_Y(a, b)$, whereas $d_2(f_n, f) \leq d_X(x_{2n-1}, x_{2n})$. Since $\{x_n\}$ is a sequence of paired isolated points, $\{f_n\}$ d_2 -converges to f . We have shown that d_1 and d_2 are not equivalent, a contradiction.

Second, suppose $\{x_n\}$ is a sequence of limit points of X with no cluster point. For each n choose ε_n such that $0 < \varepsilon_n < 1/2n$ and the collection $\{S_{\varepsilon_n}[x_n]: n \in \mathbb{Z}^+\}$ is pairwise disjoint. By assumption, there exists a continuous function $\phi: [0, 1] \rightarrow Y$ for which $\phi(0) \neq \phi(1)$; denote $\phi(0)$ by a and $\phi(1)$ by b . By the last lemma, for

each $n \in Z^+$, there exist continuous functions f_n and g_n from X to Y such that (i) both $f_n(S_{\varepsilon_n}[x_n])$ and $g_n(S_{\varepsilon_n}[x_n])$ are ε_n -dense in $\phi([0, 1])$, (ii) $g_n(z) = f_n(z) = a$ whenever $d_X(z, x_n) \geq \varepsilon_n$, (iii) $g_n(z) \neq a$ implies $f_n(z) = a$. Define $h: X \rightarrow Y$ by

$$h(z) = \begin{cases} f_n(z) & \text{if for some } n, d_X(z, x_n) < \varepsilon_n, \\ a & \text{otherwise.} \end{cases}$$

Since $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and $\{x_n\}$ has no cluster point, the collection

$$\{\overline{S_{\varepsilon_n}[x_n]} : n \in Z^+\} \cup \left\{ \bigcap_{n=1}^{\infty} \{z : d_X(z, x_n) \geq \varepsilon_n\} \right\}$$

is a locally finite closed cover of X . Since the restriction of h to each set in the cover is continuous, the amalgamation is continuous [4]. For exactly the same reasons, for each $n \in Z^+$, the function $h_n: X \rightarrow Y$ defined by

$$h_n(z) = \begin{cases} g_n(z) & \text{if } d_X(z, x_n) < \varepsilon_n, \\ h(z) & \text{otherwise,} \end{cases}$$

is continuous. We claim that $d_2(h_n, h) \leq 1/n$, more precisely, that $S_{1/n}[h] \supset h_n$ and $S_{1/n}[h_n] \supset h$. We verify just the first inclusion, leaving the second to the reader. If $d_X(z, x_n) \geq \varepsilon_n$, then $(z, h_n(z))$ actually lies on the graph of h . If $d_X(z, x_n) < \varepsilon_n$, then by condition (i) $h_n(z) = g_n(z)$ lies in $\phi([0, 1])$. Also by condition (i) there exists $w \in S_{\varepsilon_n}[x_n]$ for which $d_Y(f_n(w), g_n(z)) < \varepsilon_n$. As a result,

$$\begin{aligned} \rho[(z, h_n(z)), (w, h(w))] &= \rho[(z, g_n(z)), (w, f_n(w))] \\ &< \max\{2\varepsilon_n, \varepsilon_n\} < 1/n. \end{aligned}$$

Again, $(z, h_n(z)) \in S_{1/n}[h]$, and it follows that $h_n \subset S_{1/n}[h]$.

Finally, let n be any integer satisfying $1/n < \frac{1}{2} d_Y(a, b)$. By (i) there exists z in $S_{\varepsilon_n}[x_n]$ for which $d_Y(g_n(z), b) < 1/n$. By condition (iii) we conclude that $f_n(z) = a$, whence $d_Y(h_n(z), h(z)) > \frac{1}{2} d_Y(a, b)$. Thus, although $\{h_n\}$ d_2 -converges to h , it does not uniformly converge to h .

It should be observed that we do need some additional assumption on Y to guarantee that $C(X, Y)$ is sufficiently rich for there to be any hope that the equivalence of d_1 and d_2 on $C(X, Y)$ forces X to be an Atsugi space. For example, if Y is totally disconnected and X is connected, then each element of $C(X, Y)$ is uniformly continuous, and Hausdorff distance in $C(X, Y)$ yields the topology of uniform convergence.

Atsugi spaces may also be characterized in terms of a relationship between the Hausdorff metric topology and the Vietoris topology [4] on the hyperspace $\text{CL}(X)$ of closed nonempty subsets of X .

THEOREM 2. *The metric space $\langle X, d \rangle$ is an Atsugi space if and only if the Hausdorff metric topology on $\text{CL}(X)$ is at least as strong as the Vietoris topology.*

PROOF. We use yet another internal characterization of Atsugi spaces: X is Atsugi if and only if whenever A and B are disjoint closed sets, there exists $\varepsilon > 0$ for which $S_\varepsilon[A] \cap S_\varepsilon[B] = \emptyset$ [1]. Now a subbase for the Vietoris topology consists of all sets of the form $\Omega_V = \{A : A \in \text{CL}(X), A \cap V \neq \emptyset\}$ and $\Sigma_V = \{A : A \in \text{CL}(X), A \subset V\}$ where V is open in X . Suppose X is an Atsugi space. Without this assumption, each Ω_V is h_d -open. If $A \in \Sigma_V$, choose $\varepsilon > 0$ for which

$S_\varepsilon[A] \cap S_\varepsilon[V^c] = \emptyset$. Clearly, $h_d(A, B) < \varepsilon$ implies $B \in \Sigma_V$. Conversely, suppose that the h_d -topology is at least as strong as the Vietoris topology. If A and B are disjoint closed sets, there exists $\delta > 0$ for which $h_d(F, A) < \delta$ implies $F \in \Sigma_{B^c}$. Since $h_d(S_{\delta/2}[A], A) < \delta$, we obtain $S_{\delta/4}[A] \cap S_{\delta/4}[B] = \emptyset$.

If X is a complete metric space and $K(X)$ is the collection of nonempty compact subsets of X , then $K(X)$ is a closed subset of $\langle \text{CL}(X), h_d \rangle$ [3]. One would expect that the collection of nonempty Atsui subsets of X also forms a closed subset of the hyperspace. This fails; in fact, for arbitrary X , the Atsui subsets are dense in $\text{CL}(X)$. To see this, let $F \in \text{CL}(X)$ be arbitrary. By Zorn's Lemma, for each $\varepsilon > 0$, the collection

$$\{E: E \subset F \text{ and } d(x, y) \geq \varepsilon \text{ for each } \{x, y\} \subset E\}$$

has a maximal element $E(\varepsilon)$. Since $E(\varepsilon)$ is uniformly discrete, $E(\varepsilon)$ is an Atsui subset of X , and the maximality of $E(\varepsilon)$ yields $h_d(E(\varepsilon), F) \leq \varepsilon$.

Our last result answers a basic question: which metric spaces have Atsui hyperspaces?

THEOREM 3. *The hyperspace $\langle \text{CL}(X), h_d \rangle$ is an Atsui space if and only if either X is compact or X is uniformly discrete.*

PROOF. It is well known that $\langle \text{CL}(X), h_d \rangle$ is compact whenever X is compact [3]. If for some $\varepsilon > 0$ we have $d(x_1, x_2) > \varepsilon$ whenever $\{x_1, x_2\} \subset X$, then $h_d(F_1, F_2) > \varepsilon$ whenever $\{F_1, F_2\} \subset \text{CL}(X)$. In either case, $\text{CL}(X)$ is an Atsui space. To prove the converse, we assume that $\text{CL}(X)$ is an Atsui space, but X is neither compact nor uniformly discrete, and reach a contradiction. Since $\text{CL}(X)$ is an Atsui space, and the singleton subsets of X form a closed subset of the hyperspace, the metric space X must be Atsui. Since X is not uniformly discrete, there is a point p in X' , and since X is noncompact, there is a sequence $\{x_n\}$ in X with no cluster point. For each $n \in \mathbb{Z}^+$ the set $\{x_n, p\}$ is a limit point of $\text{CL}(X)$. However, the sequence $\{x_1, p\}, \{x_2, p\}, \{x_3, p\}, \dots$ can have no cluster point in $\text{CL}(X)$, or else $\{x_n\}$ would have a cluster point in X . This violates the compactness of the set of limit points of $\text{CL}(X)$.

We remark in closing that Theorem 3 remains valid if we replace $\text{CL}(X)$ by $K(X)$ in its statement.

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