PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 4, December 1985

SHORTER NOTES

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A NOTE ON MODULE HOMOTOPY AND CHAIN HOMOTOPY

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ABSTRACT. The Eckmann-Hilton (injective) homotopy category of modules over a ring Λ is equivalent to a certain full subcategory of the cochain homotopy category of cochain complexes of modules over Λ.

0. Introduction. Eckmann [**1**] and Hilton [**2, 3**] reported separately the results of their joint work on a homotopy theory of modules, constructed by analogy with the homotopy theory of pointed topological spaces. In retrospect it is perhaps surprising that the simple result reported here appears not to have been noticed, although the constituent facts have long been well known, and the proof is obvious.

The result may be interpreted this way. In topology, homology groups are homotopy invariant—that is, the homomorphism $f_*\colon HX \to HY$ induced by a continuous function $f\colon X \to Y$ depends only on the homotopy class of f. If the analogy referred to above has validity, surely a similar statement should be true in module theory. Corollary 2 below asserts that this is so, provided a small modification is made in dimension 0.

1. An equivalence of categories. Let $\underline{A}$ be an abelian category with sufficiently many injectives, and let $\underline{HA}$ denote the injective homotopy category of $\underline{A}$ [**2**]. Recall that two morphisms $\phi, \psi\colon A \to A'$ in $\underline{A}$ are (injectively) homotopic, or i-homotopic, if $\phi - \psi$ extends to some injective container of A. Of course, $\phi - \psi$ then extends to *any* container of A. We will simply write $\phi \sim \psi$ for the relation of i-homotopy.

Now let $\underline{\text{IRA}}$ be the category whose objects are injective resolutions of objects of $\underline{A}$ and whose arrows are morphisms of exact sequences:

$$\begin{array}{ccccccc}
A & \overset{d}{\rightarrowtail} & I_0 & \overset{d_0}{\rightarrow} & I_1 & \overset{d_1}{\rightarrow} & \cdots \\
\phi\downarrow & & \downarrow\phi^0 & & \downarrow\phi^1 & & \\
A' & \overset{d'}{\rightarrowtail} & I_0' & \overset{d_0'}{\rightarrow} & I_1' & \overset{d_1'}{\rightarrow} & \cdots
\end{array}$$

Received by the editors November 2, 1984 and, in revised form, January 30, 1985.

1980 *Mathematics Subject Classification*. Primary 18G35, 55U35; Secondary 18E10, 55U15.

Key words and phrases. Chain homotopy, injective homotopy, projective homotopy, homotopy of modules, injective resolution, projective resolution, abstract homotopy, abelian category.

[1]The first author acknowledges grants to the Topology Research Group from the University of Cape Town and the South African Council for Scientific and Industrial Research.

We point out that, in the theory of derived functors, one regards $\{I_n, d_n;\ n \geqslant 0\}$ as the injective resolution; here, however, we preserve the embedding $A \rightarrowtail I_0$ as part of the resolution and, thus, as part of the object in $\underline{\mathrm{IRA}}$.

Now, since $\underline{\mathrm{IRA}}$ is a full subcategory of the category of cochain complexes of $\underline{A}$, the notion of cochain homotopy is available in $\underline{\mathrm{IRA}}$. It has long been known that ϕ determines the cochain map $\phi^{\cdot}$ up to cochain homotopy—indeed, this is the very basis of the theory of derived functors. However, the following refinement of this remark seems to have escaped attention.

Let $\underline{\mathrm{HIRA}}$ denote the (cochain) homotopy category of $\underline{\mathrm{IRA}}$. We have a forgetful functor F: $\underline{\mathrm{IRA}} \to \underline{A}$ given by $F(A; d_{\bullet}) = A$, $F(\phi; \phi^{\cdot}) = \phi$.

THEOREM 1. *The functor F induces a functor HF: $\underline{HIRA} \to \underline{HA}$ which is an equivalence of categories.*

PROOF. A cochain homotopy $\Gamma = (\alpha^0, \alpha^1, \dots)$: $(\phi; \phi^{\cdot}) \simeq (\psi; \psi^{\cdot})$ is a collection of morphisms α^0: $I_0 \to A'$, α^n: $I_n \to I'_{n-1}$, $n \geqslant 1$, such that

$$(1.1)\qquad \alpha^0 d = \phi - \psi, \quad \alpha^{n+1} d_n + d'_{n-1}\alpha^n = \phi^n - \psi^n, \qquad n \geqslant 0\ \left(d'_{-1} = d'\right).$$

$$\begin{array}{ccccccccccccc}
A & \overset{d}{\rightarrowtail} & I_0 & \overset{d_0}{\to} & \cdots & \overset{d_{n-1}}{\to} & I_n & \overset{d_n}{\to} & I_{n+1} & \overset{d_{n+1}}{\to} & \cdots \\
\phi \downarrow \psi & \alpha^0 \swarrow & \phi^0 \downarrow \psi^0 & \alpha^1 \swarrow & & & \phi^n \downarrow \psi^n & \alpha^{n+1} \swarrow & \phi^{n+1} \downarrow \psi^{n+1} & & \\
A' & \overset{d'}{\rightarrowtail} & I'_0 & \overset{d'_0}{\to} & \cdots & \overset{d'_{n-1}}{\to} & I'_n & \overset{d'_n}{\to} & I'_{n+1} & \overset{d_{n+1}}{\to} & \cdots
\end{array}$$

Obviously, (1.1) implies that $\phi \sim \psi$, so that F induces HF: $\underline{\mathrm{HIRA}} \to \underline{\mathrm{HA}}$. Since $\underline{A}$ has sufficient injectives, HF is representative. Since F is full, HF is full. It remains to show that HF is faithful.

Thus we suppose that $\phi \sim \psi$: $A \to A'$. Then we can certainly find α^0: $I_0 \to A'$ so that $\alpha^0 d = \phi - \psi$ (see (1.1), (1.2)). Now factor d_0 as $I_0 \overset{\varepsilon}{\twoheadrightarrow} K_1 \overset{\mu}{\rightarrowtail} I_1$. Since

$$\left(\phi^0 - \psi^0 - d'\alpha^0\right)d = d'\left(\phi - \psi - \alpha^0 d\right) = 0,$$

it follows by exactness that $\exists\ \lambda^1$: $K_1 \to I'_0$ such that $\lambda^1\varepsilon = \phi^0 - \psi^0 - d'\alpha^0$. Since I'_0 is injective, $\exists\ \alpha^1$: $I_1 \to I'_0$ with $\alpha^1\mu = \lambda^1$, so $\alpha^1 d_0 = \phi^0 - \psi^0 - d'\alpha^0$ or $\alpha^1 d_0 + d'\alpha^0 = \phi^0 - \psi^0$ (see (1.1), (1.2)). It is now obvious how we proceed inductively to establish the morphisms α^n: $I_n \to I'_{n-1}$, $n \geqslant 1$, to satisfy (1.1); we simply use the exactness of the top row of (1.2) and the fact that the bottom row of (1.2) is a cochain complex consisting of injective objects from I'_0 onward. The proof of Theorem 1 is complete.

We may draw the following consequences from our proof of Theorem 1.

COROLLARY 2. *Let $\underline{T}$: $\underline{A} \to \underline{B}$ be an additive functor of abelian categories, and let $\underline{A}$ have enough injectives (projectives). Then injectively (projectively) homotopic morphisms ϕ, ψ: $A \to A'$ in $\underline{A}$ induce the same morphisms under right (left) derived functors, $R^nT\phi = R^nT\psi$ $(L_nT\phi = L_nT\psi)$, where $n \geqslant 1$.*

Note that we also get homotopy invariance 'in dimension 0' if we preserve the embedding $A \rightarrowtail I_0$ as part of the resolution, so that R^0TA is redefined as the homology of $TA \to TI_0 \to TI_1$. Of course, this new definition yields a quotient of the traditional R^0TA. A similar remark applies to L_0TA.

References

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