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ON THE STRUCTURE OF VN(G) FOR ALMOST CONNECTED GROUPS

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ABSTRACT. We prove that for any almost connected locally compact group G, the von Neumann algebra VN(G), generated by the left regular representation of G is semifinite.

Let G be a locally compact group and VN(G) the von Neumann algebra generated by the left regular representation λ of G on $L^2(G)$. As a result of the work of Dixmier in 1969 [**1**] and Pukanszky in 1972 [**2**], it was proved that VN(G) is semifinite if G is connected. The purpose of this note is to use this result and simple arguments to prove that VN(G) is semifinite if G is almost connected.

THEOREM. *If G is an almost connected locally compact group, then* VN(G) *is semifinite.*

PROOF. First suppose that G is a Lie group with H its component of identity. Then H is an open normal subgroup of finite index, say n, in G, and $L^2(G)$ can be considered as a direct sum of n copies of $L^2(H)$. When VN(G) is represented as an algebra of $n \times n$ matrices on $L^2(H)$, the matrix entries are found to come from VN(H). Therefore we can identify each $T \in \mathrm{VN}(G)$ by the matrix $(T_{i,j})$, with $T_{i,j} \in \mathrm{VN}(H)$ for all $i, j \leqslant n$. Let Φ be the function from $\mathrm{VN}(G)^+$ into $[0, \infty]$ given by $\Phi((T_{i,j})) = \sum_{k=1}^{n} \phi(T_{k,k})$, where ϕ is a faithful normal semifinite trace on VN(H). Then it is easy to check that Φ is a faithful normal semifinite trace on VN(G), which implies that VN(G) is semifinite.

Now suppose G is an almost connected group. Then every neighborhood of the identity contains a compact normal subgroup K such that G/K is an almost connected Lie group. Therefore VN(G/K) is semifinite. Let $E_K = \lambda(\mu_K)$ be the central projection in VN(G) corresponding to K as defined in [**3**]. (μ_K is normalized Haar measure on K extended trivially to all of G.) Then by Proposition (3.6) of [**3**], $E_K\mathrm{VN}(G) \simeq \mathrm{VN}(G/K)$, which implies that E_K is a semifinite central projection. Next choose a basis $\{U_i\}$ of the identity in G directed by inclusion, and let K_i be a compact normal subgroup contained in U_i such that G/K_i is an almost connected Lie group. Then $\lambda(\mu_{K_i})$ converges to the identity in the weak operator topology. Hence VN(G) is semifinite. Q.E.D

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