WARING'S PROBLEM FOR FINITE INTERVALS

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ABSTRACT. Let $f(n, k, s)$ denote the cardinality of the smallest set A of nonnegative k -th powers such that every integer in $[0, n]$ is a sum of s elements of A , and let $\beta(k, s) = \limsup_{n \rightarrow \infty} \log f(n, k, s) / \log n$. Clearly, $\beta(k, s) \geq 1/s$. In this paper it is proved that $f(n, k, s) < cn^{1/s - g(k) + k}$ for all $n \geq n_1(k, s)$, where $g(k)$ is defined as in Waring's problem, and $\beta(k, s) \sim 1/s$ as $s \rightarrow \infty$.

Lagrange proved that every nonnegative integer is the sum of four squares. Wieferich proved that every nonnegative integer is the sum of nine cubes, and Linnik showed that every sufficiently large integer is the sum of seven cubes. For $k \geq 2$, let $g(k)$ (resp. $G(k)$) denote the smallest number such that every (resp. every sufficiently large) nonnegative integer is the sum of $g(k)$ (resp. $G(k)$) nonnegative k th powers. Hilbert proved that $g(k)$ and $G(k)$ exist.

In general, if A is a set of nonnegative integers, let sA denote the set of all sums of s not necessarily distinct elements of A . If $n \in sA$ for all (resp. all sufficiently large) nonnegative integers n , then A is a basis (resp. asymptotic basis) of order s .

Define the counting function $A(x)$ of the set A by

$$A(x) = \sum_{\substack{a \in A \\ 1 \leq a \leq x}} 1.$$

If A is a basis or asymptotic basis of order s , then $A(x) > c_0 x^{1/s}$ for some $c_0 > 0$ and all $x \geq x_0$. Raikov [5] and Stöhr [6] constructed "thin" bases of order s such that $A(x) < c_1 x^{1/s}$, and Cassels [1] constructed a basis $A = \{a_k\}_{k=0}^{\infty}$ of order s such that $a_k \sim \alpha k^s + O(k^{s-1})$ for some $\alpha > 0$. The sets constructed by Cassels, Raikov, and Stöhr are highly artificial. It is more interesting to prove the existence of thin bases among the classical sequences of additive number theory.

Some results are known for k th powers and primes. Using a probabilistic method, Erdős and Nathanson [3] showed that for every $\epsilon > 0$ there exists a set A of squares such that (i) every nonnegative integer is the sum of four squares in A , (ii) if $n \neq 4^t(8k+7)$, then n is the sum of three squares in A ; and (iii) $A(x) \sim c_2 x^{(1/3)+\epsilon}$. They also proved that for $\epsilon > 0$ there exists a set A of numbers of the form p and pq , where p and q are primes, such that (i) every sufficiently large even integer is the sum of two numbers in A , and (ii) $A(x) \sim c_3 x^{(1/2)+\epsilon}$. Nathanson [4] proved that if $\epsilon > 0$ and $s > s_0(k)$, there exists a sequence A of k th powers such that (i) A is a basis of order s , and (ii) $A(x) \sim x^{(1/k)(1-1/s)+\epsilon}$.

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If A is a finite set of nonnegative integers, let $\text{card}(A)$ denote the cardinality of A . Let $[a, b]$ denote the set of integers m such that $a \leq m \leq b$. The set A is a *basis of order s for n* if $[0, n] \subseteq sA$. Clearly, $\text{card}(A) > c_0 n^{1/s}$ if A is a basis of order s for n . Choi, Erdős, and Nathanson [2] constructed a finite set A of squares such that $[0, n] \subseteq 4A$ and $\text{card}(A) < (4/\log 2)n^{1/3} \log n$. They conjectured that for every $\varepsilon > 0$ there exists a basis A of squares of order 4 for n such that $\text{card}(A) < cn^{(1/4)+\varepsilon}$. Zöllner [7] has recently proved this conjecture. It is not known if the exponent can be reduced from $(1/4) + \varepsilon$ to $1/4$.

Much less is understood about sums of k th powers. Nathanson [4] proved that if $s > g(k)$ and $n \geq n(\varepsilon)$, there exists a set A of k th powers such that $[0, n] \subseteq sA$ and $\text{card}(A) < (2 + \varepsilon)n^{1/(k+1)}$. For $s > s_0(k) + 1$, the exponent can be reduced to $\gamma/(k + \gamma)$, where $\gamma = 1 - (1/s) + \varepsilon$.

Let $f(n, k, s)$ denote the cardinality of the smallest set A of k th powers such that $[0, n] \subseteq sA$. Define

$$\beta(k, s) = \limsup_{n \rightarrow \infty} \frac{\log f(n, k, s)}{\log n}.$$

Clearly, $\beta(k, s) \geq 1/s$. In this paper I prove that $\beta(k, s) \leq 1/(s - g(k) + k)$ for all $s \geq g(k)$, and so $\beta(k, s) \sim 1/s$. This answers a question posed in [4].

LEMMA 1. *Let $k \geq 2$ and $n \geq 1$. Let $\lambda, \mu \in \mathbb{R}$ with $0 < \lambda < \mu \leq 1$. Then there is a set B of positive k th powers such that (i) $\text{card}(B) \leq 2n^{\mu-\lambda}$, and (ii) if $m \in [n^\lambda, 2n^\mu]$ there is $b^k \in B$ such that*

$$0 \leq m - b^k < n^\lambda + 2^{k+1}n^{\mu(1-(1/k))}.$$

PROOF. Let $[x]$ denote the integer part of x . Define the set B as follows:

$$B = \left\{ \left[q^{1/k} n^{\lambda/k} \right]^k \mid q \in [1, 2n^{\mu-\lambda}] \right\}.$$

Let $m \in [n^\lambda, 2n^\mu]$. Let $q = [m/n^\lambda]$. Then $q \in [1, 2n^{\mu-\lambda}]$ and $b^k \in B$, where $b = [q^{1/k} n^{\lambda/k}]$. Clearly, $b^k \leq qn^\lambda \leq m$, and

$$\begin{aligned} b^k &> (q^{1/k} n^{\lambda/k} - 1)^k > qn^\lambda - 2^k (qn^\lambda)^{(k-1)/k} \\ &> m - n^\lambda - 2^k (2n^\mu)^{1-(1/k)}. \end{aligned}$$

This proves Lemma 1.

LEMMA 2. *Let $2 \leq k < t$ and let $i \in [1, t - k]$. For $n \geq n_0(k, t)$ there is a set B_i of positive k th powers such that (i) $\text{card}(B_i) \leq 2n^{1/t}$ and (ii) if $m \in [n^{(k+i-1)/t}, 2n^{(k+i)/t}]$, there is $b^k \in B_i$ such that*

$$0 \leq m - b^k < 2n^{(k+i-1)/t}.$$

PROOF. Apply Lemma 2 with $\lambda = (k + i - 1)/t$ and $\mu = (k + i)/t$. Since $\mu - \lambda = 1/t$ and

$$\begin{aligned} n^\lambda + 2^{k+1}n^{\mu(1-(1/k))} &= n^{(k+i-1)/t} (1 + 2^{k+1}n^{-i/tk}) \\ &\leq 2n^{(k+i-1)/t} \end{aligned}$$

for $n \geq 2^{tk(k+1)} = n_0(k, t)$, the proof of Lemma 2 is complete.

THEOREM. *Let $k \geq 2$ and $s \geq g(k)$. For $n \geq n_1(k, s)$ there is a set A of k th powers such that A is a basis of order s for n and*

$$\text{card}(A) < 2(s - g(k) + 1)n^{1/\Lambda(s-g(k)+k)}.$$

PROOF. Let $t = s - g(k) + k$ and $n_1(k, s) = n_0(k, t)$. Let $B_0 = \{b^k \mid 0 \leq b \leq (2n)^{1/t}\}$ and let B_i be the set defined in Lemma 2 for $i = 1, 2, \dots, t - k$. Let $A = \bigcup_{i=0}^{t-k} B_i$. Then $[n^{1/t}]^k \in B_0 \cap B_1$ and so

$$\text{card}(A) < 2(t - k + 1)n^{1/t} = 2(s - g(k) + 1)n^{1/\Lambda(s-g(k)+k)}.$$

Let $I_0 = [0, 2n^{k/t}]$ and $I_i = [n^{(k+i-1)/t}, 2n^{(k+i)/t}]$ for $i \in [1, t - k]$. Then $[0, n] \subseteq \bigcup_{i=0}^{t-k} I_i$.

Let $m \in [0, n]$. If $m \in I_0$, then $m \in g(k)B_0 \subseteq sA$. If $m \in I_j$ for some $j \in [1, t - k]$, then by Lemma 2 there is a k th power $b^k \in B_j$ such that $m - b^k \in \bigcup_{i=0}^{j-1} I_i$. It follows that there exist $b_1^k, \dots, b_h^k \in \bigcup_{i=1}^{t-k} B_i$ for some $h \in [1, t - k]$ such that $m - (b_1^k + \dots + b_h^k) \in I_0$, and so $m \in (h + g(k))A \subseteq sA$. This proves the Theorem.

COROLLARY 1. *Let $k \geq 2$ and $\epsilon > 0$. For $s \geq s(k, \epsilon)$ and $n \geq n_2(k, s)$ there exists a set A of k th powers such that A is a basis of order s for n and $\text{card}(A) < n^{(1/s)+\epsilon}$.*

COROLLARY 2. *Let $k \geq 2$. Then $\beta(k, s) \sim 1/s$ as $s \rightarrow \infty$.*

PROOF. The Theorem implies that

$$1/s \leq \beta(k, s) \leq 1/(s - g(k) + k)$$

and so $\lim_{s \rightarrow \infty} s\beta(k, s) = 1$.

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