THE UNIVERSALITY OF WORDS $x^r y^s$ IN ALTERNATING GROUPS

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ABSTRACT. If r, s are nonzero integers and m is the largest squarefree divisor of rs , then for every element z in the alternating group A_n , the equation $z = x^r y^s$ has a solution with $x, y \in A_n$, provided that $n \geq 5$ and $n \geq (5/2)\log m$. The bound $(5/2)\log m$ improves the bound $4m + 1$ of Droste. If $n \geq 29$, the coefficient $5/2$ may be replaced by 2; however, $5/2$ cannot be replaced by 1 even for all large n .

1. Introduction. For a group G , a word $W(x_1, \dots, x_k)$ in free variables $x_1, \dots, x_k$ is said to be G -universal if $G \subset W(G, \dots, G)$, i.e., if for every $g \in G$, there exist $g_1, \dots, g_k \in G$ such that $g = W(g_1, \dots, g_k)$. Let A_n denote the alternating group contained in the symmetric group S_n on $\{1, \dots, n\}$. For each pair of nonzero integers r, s , let $m = m(r, s)$ denote the product of the distinct prime factors of rs . It is known [6, Theorem 1; 9] that the word $x^r y^s$ is A_n -universal for all $n \geq 4m + 1$. In Theorem 3, we show that the condition $n \geq 4m + 1$ may be replaced by the condition $n \geq (5/2)\log m$ if $n \geq 5$, and even by the condition $n \geq 2\log m$ if $n \geq 29$. Cases $n < 29$ are treated separately in Theorem 2. Theorem 1 is used to show that Theorem 2 is "best possible". In Theorem 3', we show that the bound $2\log m$ for $n \geq 29$ cannot be replaced by $\log m$, even just for $n \geq N_0$; however, $2\log m$ can be replaced by $C\log m$ for any constant $C > 8/5$, provided that $n \geq N_0(C)$.

2. Statements of theorems.

THEOREM 1. Let n, a, b be positive integers with $n \geq 7$ and $a + b < 2[3n/4]$, where $[x]$ denotes the integer part of x . If $n \equiv 0$ or $1 \pmod{4}$, let w be any product of $2[n/4]$ disjoint 2-cycles in S_n , and if $n \equiv 3 - \epsilon \pmod{4}$ with $\epsilon = 0$ or 1 , let w be any product of $2[n/4] - \epsilon$ disjoint 2-cycles with a disjoint $(3 + \epsilon)$ -cycle in S_n . Then w does not equal a product of an a -cycle and a b -cycle in S_n .

REMARK. Theorem 1 is best possible in the sense that, for each n , the symbol $<$ cannot be replaced by $\leq$. For, if $a = b = [3n/4]$, then by [1, or 3, Corollary 2.10], every element of A_n is a product of two b -cycles in S_n .

THEOREM 2. Let P_n denote the product of the distinct primes $\leq n$. For each $n \leq 28$, the word $x^r y^s$ is A_n -universal when $m < P_n/d_n$, where the values of d_n are given in

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the following table:

n	1,2	3,4	5,6,7,10,15	8,9,14	11,12,13	16,17,18	19,20,21	22,23	24,25	26	27,28
d_n	$0 +$	2	1	5	7	11	$11 \cdot 13$	13	$13 \cdot 17$	17	$17 \cdot 19$

REMARK. Theorem 2 is best possible in the sense that, for each n , the symbol $<$ cannot be replaced by $\leq$. To see this, first suppose that $n = 3$ or 4. Then x^3y^3 is a word with $m = 3 = P_n/2$ which is not A_n -universal, since the 3-cycle (123) does not have the form x^3y^3 . Next suppose that $n = 5, 6, 7, 10$, or 15. Then $x^{n!}y^{n!}$ is a word with $m = P_n/1$ which is not A_n -universal since $x^{n!}$ is trivial for all $x \in A_n$. If $n = 8$ or 9, then $x^{n!/5}y^{n!/5}$ is a word with $m = P_n/5$ which is not A_n -universal since, by Theorem 1 with $a = b = 5$, (12)(34)(56)(78) is not the product of two 5-cycles. The values of n in the ranges 11–13, 16–28 may be handled similarly. For example, if $n = 19$, then $x^{n!/143}y^{n!/143}$ is a word with $m = P_n/143$ which is not A_n -universal since, by Theorem 1, (12)(34)(56)(78)(9 10)(11 12)(13 14)(15 16)(17 18 19) is not the product of two 13-cycles nor two 11-cycles nor an 11-cycle times a 13-cycle. (Note that there is no element of order 143 in A_{19} .) Finally, suppose that $n = 14$. Then $x^{n!/25}y^{n!/25}$ is a word with $m = P_n/5$ which is not A_n -universal, for it is known [4] that (12)(34)(56)(78)(9 10)(11 12 13 14) is not the product of two elements of order 5 in A_{14} . (It is stated incorrectly in [5, p. 39] that for $n > 11$, every element of A_n is the product of two elements of order 5.)

THEOREM 3. *The word x^ry^s is A_n -universal for all $n \geq (5/2)\log m$ if $n \geq 5$. If $n \geq 29$, then x^ry^s is A_n -universal for all $n \geq 2\log m$.*

THEOREM 3'. *Let C be any constant exceeding $8/5$. For all $n \geq N_0(C)$, x^ry^s is A_n -universal whenever $n \geq C\log m$. On the other hand, it is not true that, for all $n \geq N_0$, every word x^ry^s is A_n -universal whenever $n \geq \log m$.*

3. Lemmas.

LEMMA 4. *Choose a positive integer b such that $[3n/4] \leq b \leq n$. Then every element of A_n is a product of two b -cycles in S_n .*

PROOF. This is easily checked for $n \leq 4$, and for $n \geq 5$, it follows from [3, Corollary 2.10].

LEMMA 5. *Choose integers $u, v \geq 4$ such that $[3n/4] + 1 \leq u + v \leq n$. Then every element of A_n is a product of two words, each of which is a product of a u -cycle and a disjoint v -cycle in S_n .*

PROOF. This follows from the proof of [3, Corollary 2.10] and from the theorem in [3, p. 168].

REMARK. On lines 13, 17, 19, 20 of [3, p. 168], replace misprints $q = 3, 4.07, l$, and η by $q - 3, 4.09, |l|$, and ηe , respectively.

LEMMA 6. Let $n \geq 5$, and choose an integer v such that $[3n/4] - 1 \leq v \leq n - 2$. Then every element of A_n is a product of two words, each of which is a product of a 2-cycle and a disjoint v -cycle in S_n .

PROOF. Apply [3, Theorem 3.02] and the proof of [3, Corollary 2.10].

LEMMA 7. If every element of A_n is a product of two words in A_n , each of whose orders is prime to m , then $x^r y^s$ is A_n -universal.

PROOF. Given $z \in A_n$, write $z = w_1 w_2$, where $w_i \in A_n$ has order e_i and $(m, e_i) = 1$ for $i = 1, 2$. Define R, S by $Rr \equiv 1 \pmod{e_1}$, $Ss \equiv 1 \pmod{e_2}$. Then $z = x^r y^s$ where $x = w_1^R, y = w_2^S$.

LEMMA 8. Suppose that $2 \nmid m$. Then $x^r y^s$ is A_n -universal for all $n \geq 5$.

PROOF. Assume that $x^r y^s$ is not A_n -universal. In view of Lemma 7, the desired contradiction will be obtained if one can apply Lemma 5 or 6 to show that every element of A_n is a product of two nontrivial words in A_n , each of order a power of 2. If $n = 5, 6$, or 7 , apply Lemma 6 with $v = 2, 4$, or 4 , respectively. Thus assume that $n \geq 8$. Define the integer c by $n/2 \leq 2^c < n$, and choose the largest integer d such that $2^c + 2^d \leq n$. If $d = 0$, then $n = 2^c + 1$, so apply Lemma 5 with $u = v = 2^{c-1}$. If $d = 1$, then $n = 2^c + 2$ or $2^c + 3$, so apply Lemma 6 with $v = 2^c$. If $d > 1$, we can apply Lemma 5 with $u = 2^c, v = 2^d$; to see that $u + v = 2^c + 2^d > 3n/4$, note that by definition of d , $2^{d+1} + 2^c > n$, so $2(2^c + 2^d) > n + 2^c \geq 3n/2$.

LEMMA 9. If $3 \nmid m$, then $x^r y^s$ is A_n -universal for all $n \geq 1$.

PROOF. This follows from [7, Proposition 2].

REMARK. An analogue of Lemmas 8 and 9 with the condition $5 \nmid m$ is given in [4]. It would be interesting to find an analogue for a general prime $p \nmid m$.

Let $x = n/8$. Let $p_1 < \dots < p_\alpha$ denote the primes in the interval $(x, 2x]$, $P_1 < \dots < P_\beta$ the primes in $(5x, 6x]$, $q_1 < \dots < q_\gamma$ the primes in $(2x, 3x]$, and $Q_1 < \dots < Q_\delta$ the primes in $(4x, 5x]$.

LEMMA 10. Let $n \geq 5$. Suppose that $x^r y^s$ is not A_n -universal. Then $6 \mid m$. Also, m is divisible by each prime in $(3n/4 - 1, n]$ and each prime in $(3n/8, n/2]$. Further, for each $i = 1, 2, \dots, \min(\alpha, \beta)$, at least one of p_i, P_i divides m , and, for each $j = 1, 2, \dots, \min(\gamma, \delta)$, at least one of q_j, Q_j divides m .

PROOF. By Lemmas 8 and 9, we have $6 \mid m$. If $p \in (3n/4 - 1, n]$ is a prime ≥ 5 , then in view of Lemma 7, one can apply Lemma 4 with $b = p$ to show that $p \mid m$. If $p \in (3n/8, n/2]$ is a prime ≥ 5 , apply Lemma 5 with $u = v = p$ to see that $p \mid m$. Finally, apply Lemma 5 with $u = p_i, v = P_i$ or $u = q_j, v = Q_j$ to complete the proof.

4. Proofs of theorems.

PROOF OF THEOREM 1. This follows easily from a beautiful result of Boccara [2, Theorem 4.1].

PROOF OF THEOREM 2. Assume that $x^r y^s$ is not A_n -universal. If $n = 1$ or 2 , then A_n would be trivial, so $n \geq 3$. If $n = 3$ or 4 , then $m \geq P_n/2 = 3$, because $3 \mid m$ by Lemma 9. If n is in the range 5 – 14 , then $m \geq P_n/d_n$, since P_n/d_n divides m by

Lemma 10. If $n = 16, 17$, or 18 , then $P_n/5d_n = P_n/55$ divides m by Lemma 10. Moreover, one of $5, 11$ also divides m by Lemma 5 with $u = 5, v = 11$. Thus, $m \geq P_n/11 = P_n/d_n$ if $n = 16, 17$, or 18 , and the same type of argument shows that $m \geq P_n/d_n$ if $n = 22, 23$, or 26 . Now suppose that $n = 19, 20$, or 21 . Then $P_n/35d_n = P_n/(5 \cdot 7 \cdot 11 \cdot 13) = 19 \cdot 17 \cdot 3 \cdot 2$ divides m by Lemma 10. Moreover, 7 or 11 must divide m by Lemma 5 with $u = 7, v = 11$, and 5 or 13 must divide m by Lemma 5 with $u = 5, v = 13$. Thus $m \geq P_n/143 = P_n/d_n$ if $n = 19, 20$, or 21 , and the same type of argument shows that $m \geq P_n/d_n$ if $n = 24, 25, 27$, or 28 . Finally, suppose that $n = 15$. Then $P_n/5$ divides m by Lemma 10. It is known [4] that every element of A_{15} is a product of two elements of order 5 in A_{15} , so by Lemma 7, $x^r y^s$ would be A_{15} -universal if $5 \nmid m$. Thus $5 \mid m$. It follows that $P_n \mid m$, so $m \geq P_n = P_n/d_n$.

In the proofs below, we will use the number theoretic functions

$$\theta(x) = \sum_{p \leq x} \log p, \quad \pi(x) = \sum_{p \leq x} 1,$$

where p runs through the primes ≥ 2 .

PROOF OF THEOREM 3'. Assume that $x^r y^s$ is not A_n -universal. We will show that $\log m > n/C$ if $n \geq N_0(C)$. By Lemma 10,

$$\begin{aligned} \log m &> \theta(n) - \theta(3n/4) + \theta(n/2) - \theta(3n/8) \\ &\quad + \sum_{i=1}^{\min(\alpha, \beta)} \log p_i + \sum_{j=1}^{\min(\gamma, \delta)} \log q_j. \end{aligned}$$

Thus,

$$\begin{aligned} (1) \quad \log m &> \theta(n) - \theta(3n/4) + \theta(n/2) - \theta(3n/8) \\ &\quad + \min(\alpha, \beta) \log(n/8) + \min(\gamma, \delta) \log(n/4), \end{aligned}$$

where $\alpha = \pi(2n/8) - \pi(n/8)$, $\beta = \pi(6n/8) - \pi(5n/8)$, $\gamma = \pi(3n/8) - \pi(2n/8)$, $\delta = \pi(5n/8) - \pi(4n/8)$. Now apply the asymptotic formulas [8, p. 66] $\theta(n) \sim n$, $\pi(n) \sim n/\log n$ ($n \rightarrow \infty$). Since $8/(5C) < 1$, it follows from (1) that, for $n \geq N_0(C)$,

$$\log m > (8/(5C))(n/4 + n/8 + n/8 + n/8) = n/C.$$

This proves the first part of Theorem 3'.

Let n be any of the infinitely many integers for which $n \geq \theta(n)$ [8, p. 67]. Put $r = s = n!$. Then $\log m = \theta(n) \leq n$, yet $x^r y^s$ is not A_n -universal since x^r and y^s are trivial for all $x, y \in A_n$.

PROOF OF THEOREM 3. Assume that $x^r y^s$ is not A_n -universal. We will first show that $\log m > n/2$ if $n \geq 29$. Write $x = n/8$. Then

$$\sum_{i=1}^{\min(\alpha, \beta)} \log p_i \geq \begin{cases} \theta(2x) - \theta(x) & \text{if } \alpha \leq \beta, \\ \beta \log x & \text{if } \alpha > \beta, \end{cases}$$

and

$$\sum_{j=1}^{\min(\gamma, \delta)} \log q_j \geq \begin{cases} \theta(3x) - \theta(2x) & \text{if } \gamma \leq \delta, \\ \delta \log 2x & \text{if } \gamma > \delta. \end{cases}$$

For brevity, write $\theta(i, j) := \theta(ix) - \theta(jx)$. Then, by (1),

$$(2) \quad \log m > \begin{cases} \theta(8, 6) + \theta(4, 1) & \text{if } \alpha \leq \beta, \gamma \leq \delta, \\ \theta(8, 6) + \theta(4, 3) + \theta(2, 1) + \delta \log 2x & \text{if } \alpha \leq \beta, \gamma > \delta, \\ \theta(8, 6) + \theta(4, 2) + \beta \log x & \text{if } \alpha > \beta, \gamma \leq \delta, \\ \theta(8, 6) + \theta(4, 3) + \delta \log 2x + \beta \log x & \text{if } \alpha > \beta, \gamma > \delta. \end{cases}$$

Case 1. $n > 10^8$. By [8, Theorems 9 and 10],

$$\theta(2, 1) > (.98)(2x) - (1.02)x = .94x,$$

$$\theta(4, 3) > .86x, \quad \theta(4, 2) > 1.88x, \quad \theta(4, 1) > 2.9x,$$

$$\theta(8, 6) > 1.72x, \quad \theta(6, 5) > .78x, \quad \theta(5, 4) > .82x,$$

$$\begin{aligned} \beta \log x &= (\log x)(\pi(6x) - \pi(5x)) > \frac{\log x}{\log 6x} (\theta(6x) - \theta(5x)) \\ &> \frac{\log(10^8/8)}{\log(6 \cdot 10^8/8)} (.78x) > .7x, \end{aligned}$$

and

$$\begin{aligned} \delta \log 2x &= (\log 2x)(\pi(5x) - \pi(4x)) > \frac{\log 2x}{\log 5x} (\theta(5x) - \theta(4x)) \\ &> \frac{\log(2 \cdot 10^8/8)}{\log(5 \cdot 10^8/8)} (.82x) > .77x. \end{aligned}$$

Thus, in all four cases of (2), $\log m > 4x = n/2$.

Case 2. $7481 \leq n \leq 10^8$. By [8, Theorems 10 and 18],

$$\theta(2, 1) > (.96)(2x) - x = .92x,$$

$$\theta(4, 3) > .88x, \quad \theta(4, 2) > 1.88x, \quad \theta(4, 1) > 2.88x,$$

$$\theta(8, 6) > 1.84x, \quad \theta(6, 5) > .85x, \quad \theta(5, 4) > .85x,$$

$$\beta \log x > \frac{\log x}{\log 6x} \theta(6, 5) > \frac{\log 7481}{\log 6 \cdot 7481} (.85x) > .67x,$$

and

$$\delta \log 2x > \frac{\log 2x}{\log 5x} \theta(5, 4) > \frac{\log 2 \cdot 7481}{\log 5 \cdot 7481} (.85x) > .75x.$$

Again by (2), $\log m > 4x = n/2$.

Case 3. $223 \leq n < 7481$. It is easily checked by computer that $\log m > n/2$ as a consequence of (2) and the fact that $6|m$ (see Lemma 10).

Case 4. $29 \leq n \leq 222$. Here one proceeds as in Case 3, except that judicious use of Lemmas 5 and 10 must also be made for several values of n . We illustrate with the most troublesome value, $n = 36$. By Lemma 10, m is divisible by $2 \cdot 3 \cdot 17 \cdot 29 \cdot 31$. By Lemma 5 with $n = 36$, $u = 5$, $v = 25$, m is divisible by 5. By Lemma 5 with $u = 13$, $v = 19$, m is divisible by one of 13, 19. Similarly, m is divisible by one of 13, 23, by one of 11, 19, by one of 11, 23, and by one of 7, 23. Thus m is divisible by $7 \cdot 11 \cdot 13$ (if $23 \nmid m$) or $23 \cdot 19$ or $23 \cdot 11$. In any event, $\log m > n/2 = 18$, since $\log(2 \cdot 3 \cdot 5 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 31) > 18$.

Case 5. $5 \leq n \leq 28$. By Theorem 2, $\log m \geq \log P_n/d_n > 2n/5$, as claimed.

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