MODULES DETERMINED BY THEIR TOPS AND SOCLES

K. BONGARTZ AND S. O. SMALØ

ABSTRACT. We prove that a module in a preprojective component of the Auslander-Reiten quiver of an artin algebra is determined by its top and socle. Also other invariants determining such a module are given.

Let A be an artin algebra over a commutative artin ring k . All modules considered here will be of finite length. A connected component $\mathcal{C}$ of the Auslander-Reiten quiver (A-R-quiver) Γ_A of A is called a preprojective component if $\mathcal{C}$ contains no oriented cycles and if any indecomposable module U belonging to $\mathcal{C}$ is isomorphic to $(\text{Tr } D)^n P$ for some indecomposable projective module P and some nonnegative number n (see [5]). We prove the following result:

THEOREM. *Let M be an indecomposable module belonging to a preprojective component of the Auslander-Reiten quiver of A . Denote by $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ a minimal projective resolution of M and by I_0 an injective hull of M . Then the following conditions on an indecomposable module N are equivalent:*

- (a) $N \simeq M$.
- (b) $|\text{Hom}(Q, N)| = |\text{Hom}(Q, M)|$ for any projective module Q . (Here $|X|$ denotes the length of X as a k -module.)
- (c) N has a minimal projective resolution $P_1 \rightarrow P_0 \rightarrow N \rightarrow 0$.
- (d) P_0 is a projective cover of N and I_0 is an injective hull of N .

Results along this line have been obtained by several persons (see [2–6]). That (b) implies (a) follows from [6], but for the convenience of the reader we include an argument for this. The other two implications seem to be new even though special cases are included in [2–4].

Before we start with the proof of the theorem, we recall some facts which we use later on without comments. It is easy to see that a component $\mathcal{C}$ of the A-R-quiver Γ_A is a preprojective component if and only if $\mathcal{C}$ contains no oriented cycles and each module X in $\mathcal{C}$ has the property that $\text{Hom}(Y, X)$ is nonzero for only a finite number of indecomposable modules Y . Furthermore, if X is in a preprojective component $\mathcal{C}$ and $\text{Hom}(Y, X)$ is nonzero, then Y is a predecessor of X , that means that there is a chain $Y = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = X$ of irreducible maps (see [5, §1.3]). In particular, Y also belongs to the preprojective component $\mathcal{C}$.

Combining this observation with some general formulas of Auslander and Reiten one obtains the following interesting consequences which are crucial for our proof.

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Let M be an indecomposable module in a preprojective component.

(1) $\text{Ext}^1(M, M)$ is equal to zero because $\text{Ext}^1(M, M)$ is isomorphic to

$$D \underline{\text{Hom}}(\text{Tr } DM, M)$$

(see [1, Proposition 2.2 and line 1, p. 253]) which is included in $D \text{Hom}(\text{Tr } DM, M)$, and this is zero because there are no oriented cycles.

(2) If each indecomposable projective module P is a predecessor of M , we have that $\text{Hom}(\text{Tr } DX, A)$ is zero for each indecomposable module X with $\text{Hom}(M, X)$ nonzero. Hence, we conclude that the injective dimension of such an X is less than or equal to one by [2, Proposition 3.8(b)].

(3) If no indecomposable injective module is a predecessor of $D \text{Tr } M$ we have that $\text{Hom}(I, D \text{Tr } X)$ is zero for all injective modules I and all predecessors X of M . Therefore, by [2, Proposition 3.7], the dual of the result referred to above, the projective dimension of X is less than or equal to one for all predecessors X of M .

We fix a set $\mathcal{S}$ of representatives of isomorphism classes of simple A -modules. For each S in $\mathcal{S}$ we denote by $P(S)$ a projective cover of S and $I(S)$ an injective hull of S . For each A -module X and each S in $\mathcal{S}$ we have that $|\text{Hom}(P(S), X)|$ is equal to $|\text{Hom}(X, I(S))|$.

Finally, before starting the proof of the theorem we recall some facts about the restriction functor $\text{Hom}(P,)$ for P a projective A -module.

The support of a module M , denoted $\text{supp } M$, is the set of all $P(S)$ such that $\text{Hom}(P(S), M)$ is nonzero. Let P be equal to $\coprod_{S \in \mathcal{T}} P(S)$ for some $\mathcal{T} \subseteq \mathcal{S}$, and let B be the opposite of the ring of endomorphisms of P , $\text{End}(P)^{\text{op}}$.

(4) $\text{Hom}(P,)$ induces an equivalence between the category of A -modules X with projective presentation $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ where the indecomposable summands of P_0 and P_1 are amongst $P(S)$ with S in $\mathcal{T}$, and the B -modules.

(5) $\text{Hom}(P,)$ induces an equivalence between the category of A -modules X such that all simple summands of the top of X and the socle of X are in $\mathcal{T}$, and the B -modules. In particular, $\text{Hom}(P,)$ induces a full and faithful functor from the subcategory of A -modules X with the support of X in $\{P(S) | S \in \mathcal{T}\}$ to the B -modules. (See [2, §1, Case (a) and (b)].)

PROOF OF THE THEOREM. To show that (b) implies (a), (c) implies (a) and (d) implies (a) we prove in each case that $\text{Hom}(N, M)$ is different from zero, hence by the comments above, N belongs to the same preprojective component as M . By symmetry, we then have that $\text{Hom}(M, N)$ is nonzero and then, by the absence of oriented cycles, M and N have to be isomorphic modules.

(b) *implies* (a). We first reduce to the case where $\text{Hom}(Q, M)$ is nonzero for all indecomposable projective modules Q by using the restriction functor $R = \text{Hom}(P,)$ where P is the direct sum of the $P(S)$ in $\text{supp } M$. Then $R(M)$ is an indecomposable B -module where $B = \text{End}(P)^{\text{op}}$ by remark (5), and $\text{Hom}(Q, RM)$ is nonzero for all indecomposable projective B -modules. Further, again by the equivalence in remark (5) it is not hard to see that RM is in a preprojective component $\mathcal{C}$ of the A - R -quiver Γ_B of B . Hence, we may assume $\text{Hom}(Q, M)$ is nonzero for all indecomposable projective modules Q . Then $P(S)$ is a predecessor of M for all S in $\mathcal{S}$ and therefore

$\text{Hom}(\text{Tr } DM, A)$ is zero, which by remark (2) implies that there is an injective resolution $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0$ of M . Applying $\text{Hom}(M, _)$ to this sequence and the fact that $\text{Ext}^1(M, M)$ is zero, we get that $|\text{Hom}(M, M)| = |\text{Hom}(M, I_0)| - |\text{Hom}(M, I_1)|$. Similarly, applying $\text{Hom}(N, _)$ to the same sequence and using the fact that $|\text{Hom}(M, I)| = |\text{Hom}(N, I)|$ for all injective modules I , we get that

$$\begin{aligned} |\text{Hom}(N, M)| &= |\text{Hom}(N, I_0)| - |\text{Hom}(N, I_1)| + |\text{Ext}^1(N, M)| \\ &\geq |\text{Hom}(N, I_0)| - |\text{Hom}(N, I_1)| \\ &= |\text{Hom}(M, I_0)| - |\text{Hom}(M, I_1)| \\ &= |\text{Hom}(M, M)| \geq 1 \end{aligned}$$

which completes the proof of this implication.

(c) *implies* (a). This implication is obvious if M is a projective module. Therefore, we may assume that M is a nonprojective module. Then consider the almost split sequence

$$0 \rightarrow D \text{Tr } M \rightarrow X \rightarrow M \rightarrow 0$$

ending in M . Now let P be the direct sum of those $P(S)$ which are in the support of X . Because the top of P_1 is isomorphic to the socle of $D \text{Tr } M$ (see e.g. [1, Proposition 5.3]), all indecomposable direct summands of P_1 occur as summands of P . Denote by R the restriction functor $\text{Hom}(P, _)$ from A -modules to B -modules where $B = \text{End}(P)^{\text{op}}$. We obtain

$$RP_1 \rightarrow RP_0 \rightarrow RM \rightarrow 0 \quad \text{and} \quad RP_1 \rightarrow RP_0 \rightarrow RN \rightarrow 0$$

as the first terms of a minimal projective resolution of RM and RN , respectively. From remark (5) we have that

$$0 \rightarrow RD \text{Tr } M \rightarrow RX \rightarrow RM \rightarrow 0$$

is an almost split sequence of B -modules. Since none of the indecomposable projective B -modules belong to cycles of nonisomorphisms between indecomposable B -modules and all indecomposable projective B -modules have nonzero homomorphisms to RX , the component $\mathcal{C}$ of the A - R -quiver Γ_B of B containing RM is a preprojective component.

Further, since $\text{Hom}(Q, RX)$ is nonzero for all indecomposable projective B -modules Q , we have that $\text{Hom}(RX, I)$ is nonzero for all indecomposable injective B -modules I . Hence, $\text{Hom}(J, D \text{Tr } M)$ is zero for all indecomposable injective B -modules J and therefore by remark (3) above

$$0 \rightarrow RP_1 \rightarrow RP_0 \rightarrow RM \rightarrow 0$$

is exact.

From this reduction we may as well assume from the beginning that

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

is exact. Applying $\text{Hom}(_, M)$ to this exact sequence and the exact sequence $P_1 \rightarrow P_0 \rightarrow N \rightarrow 0$ and using the fact that $\text{Ext}^1(M, M) = 0$ we obtain that

$$\begin{aligned} |\text{Hom}(N, M)| &\geq |\text{Hom}(P_0, M)| - |\text{Hom}(P_1, M)| \\ &= |\text{Hom}(M, M)| \geq 1. \end{aligned}$$

This completes the proof of this implication.

(d) *implies* (a). Now let P be the direct sum of the $P(S)$ when $\mathcal{S}$ is isomorphic to a direct summand of either the socle of M or the top of M .

Using remark (5) we may assume that all simple modules occur as a composition factor of either the socle of M or the top of M . In the absence of cycles, S is divided into two disjoint subsets

$$\mathcal{X} = \{ T | \text{Hom}(M, T) \neq 0 \} \quad \text{and} \quad \mathcal{Y} = \{ S | \text{Hom}(S, M) \neq 0 \}.$$

By remarks (2) and (3) each T in $\mathcal{X}$ has injective dimension less than or equal to one and each S in $\mathcal{Y}$ has projective dimension less than or equal to one. Moreover, the relation $S_1 \leq S_2$ if $\text{Ext}^1(S_1, S_2)$ is nonzero, generates a partial ordering $\leq$ on $\mathcal{S}$ since all the indecomposable projective modules belong to the preprojective component $\mathcal{C}$ containing M .

First, we prove the equality

$$|\text{Hom}(P(S), M)| = |\text{Hom}(P(S), N)| \quad \text{for all } S \in \mathcal{Y}$$

by induction on $\leq$. The statement is clear for minimal S because $P(S)$ is then isomorphic to S .

Each $S \in \mathcal{Y}$ has a minimal projective resolution $0 \rightarrow \overline{P(S)} \rightarrow P(S) \rightarrow S \rightarrow 0$ where $\overline{P(S)}$ is a direct sum of $P(S')$ with $S' < S$. We obtain exact sequences

$$0 \rightarrow \text{Hom}(S, M) \rightarrow \text{Hom}(P(S), M) \rightarrow \text{Hom}(\overline{P(S)}, M) \rightarrow \text{Ext}^1(S, M)$$

and

$$0 \rightarrow \text{Hom}(S, N) \rightarrow \text{Hom}(P(S), N) \rightarrow \text{Hom}(\overline{P(S)}, N) \rightarrow \text{Ext}^1(S, N).$$

The end terms have to be zero because S , being a predecessor of M , does not lie on an oriented cycle. Induction applies.

Similarly, we show the inequality $|\text{Hom}(M, I(T))| \geq |\text{Hom}(N, I(T))|$ for any $T \in \mathcal{X}$. This is obvious if T is maximal in $\mathcal{X}$. Let $0 \rightarrow T \rightarrow I(T) \rightarrow \overline{I(T)} \rightarrow 0$ be an injective resolution of $T \in \mathcal{X}$. Then $\overline{I(T)}$ is a direct sum of some $I(T')$ satisfying $T < T'$. An easy calculation gives the following inequality:

$$\begin{aligned} |\text{Hom}(M, I(T))| &= |\text{Hom}(M, T)| + |\text{Hom}(M, \overline{I(T)})| \\ &\geq |\text{Hom}(N, T)| + |\text{Hom}(N, \overline{I(T)})| \\ &= |\text{Hom}(N, I(T))| + |\text{Ext}^1(N, T)| \\ &\geq |\text{Hom}(N, I(T))|. \end{aligned}$$

Using remark (2) once more, M has a minimal injective resolution $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0$. We apply $\text{Hom}(M,)$ and $\text{Hom}(N,)$ to it and, using that $|\text{Hom}(P(S), X)| = |\text{Hom}(X, I(S))|$ for all modules X , we get the following inequality:

$$\begin{aligned} |\text{Hom}(N, M)| &\geq |\text{Hom}(N, I_0)| - |\text{Hom}(N, I_1)| \\ &= |\text{Hom}(M, I_0)| - |\text{Hom}(N, I_1)| \\ &\geq |\text{Hom}(M, I_0)| - |\text{Hom}(M, I_1)| \\ &= |\text{Hom}(M, M)| > 0. \end{aligned}$$

The proof of the theorem is complete.

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GESAMTHOCHSCHULE WUPPERTAL, FACHBEREICH 7 MATHEMATIK, GAUSSSTR. 20 5000 WUPPERTAL,
FEDERAL REPUBLIC OF GERMANY

UNIVERSITETET I TRONDHEIM, AVH, INSTITUTT FOR MATEMATIKK OG STATISTIKK, 7055 DRAGVOLL,
NORWAY