THE DISTANCE BETWEEN THE EIGENVALUES OF HERMITIAN MATRICES

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ABSTRACT. It is shown that the minmax principle of Ky Fan leads to a quick simple derivation of a recent inequality of V. S. Sunder giving a lower bound for the spectral distance between two Hermitian matrices. This brings out a striking parallel between this result and an earlier known upper bound for the spectral distance due to L. Mirsky.

Let A be a Hermitian matrix of order n and let $\lambda_{\downarrow}(A)$ denote the vector in $\mathbf{R}^n$ whose coordinates are the eigenvalues of A arranged as $\lambda_{[1]}(A) \geq \cdots \geq \lambda_{[n]}(A)$. Let $\lambda_{(1)}(A) \leq \cdots \leq \lambda_{(n)}(A)$ be the increasing rearrangement of these eigenvalues and $\lambda_{\uparrow}(A)$ the vector with coordinates $\lambda_{(j)}(A)$, $j = 1, 2, \dots, n$. The same symbols $\lambda_{\downarrow}(A)$ and $\lambda_{\uparrow}(A)$ will also denote the diagonal matrices which have as their diagonal entries the components of the vectors $\lambda_{\downarrow}(A)$ and $\lambda_{\uparrow}(A)$, respectively. Let $\|\cdot\|$ denote any unitarily invariant norm on the space of matrices. (See [4].)

This note is concerned with the following result:

THEOREM. *Let A and B be Hermitian matrices. Then for every unitarily invariant norm we have*

$$(1) \quad \|\lambda_{\downarrow}(A) - \lambda_{\downarrow}(B)\| \leq \|A - B\| \leq \|\lambda_{\downarrow}(A) - \lambda_{\uparrow}(B)\|.$$

The first inequality in (1) appeared in a paper of Mirsky [4], who used a famous result of Lidskii and Wielandt to derive it. The second is proved in a recent paper of Sunder [5]. I give here another proof of the second inequality which has two attractive features: It is very short and it proceeds on exactly the same lines as the well-known proof of Lidskii, Wielandt and Mirsky for the first inequality. For illumination, I indicate how both inequalities follow from the same principle.

It is an easy consequence of the minmax principle of Wielandt that for any choice $1 \leq i_1 < \cdots < i_k \leq n$ of k indices we have

$$(2) \quad \sum_{j=1}^k \lambda_{[i_j]}(A+B) \leq \sum_{j=1}^k \lambda_{[j]}(A) + \sum_{j=1}^k \lambda_{[i_j]}(B)$$

for all $k = 1, 2, \dots, n$, with equality holding for $k = n$. (See [3, p. 242].)

Writing $x < y$ to mean that the vector x is majorised by the vector y in $\mathbf{R}^n$ (see [3]), we get from inequalities (2)

$$(3) \quad \lambda_{\downarrow}(A+B) - \lambda_{\downarrow}(B) < \lambda_{\downarrow}(A).$$

With a change of variables, this gives

$$\lambda_{\downarrow}(A) - \lambda_{\downarrow}(B) < \lambda_{\downarrow}(A - B).$$

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Now the first part of the Theorem follows using standard characterisations of majorisation together with properties of symmetric gauge functions and unitarily invariant norms. This is the well-known proof of Mirsky [4].

Now note that from (2) we can also conclude

$$(4) \quad \lambda_{\downarrow}(A + B) \prec \lambda_{\downarrow}(A) + \lambda_{\downarrow}(B).$$

In fact, for this conclusion the full force of (2) is not needed. It suffices to use the special case $(i_1, \dots, i_k) = (1, \dots, k)$ which is much easier to prove using the minmax principle of Ky Fan [2].

Replace B by $-B$ in (4) and note that $\lambda_{\downarrow}(-B) = -\lambda_{\uparrow}(B)$. This gives

$$\lambda_{\downarrow}(A - B) \prec \lambda_{\downarrow}(A) - \lambda_{\uparrow}(B).$$

But this implies

$$(5) \quad (|\lambda_{[1]}(A - B)|, \dots, |\lambda_{[n]}(A - B)|) \\ \prec_w (|\lambda_{[1]}(A) - \lambda_{(1)}(B)|, \dots, |\lambda_{[n]}(A) - \lambda_{(n)}(B)|)$$

where $\prec_w$ stands for weak majorisation [3, p. 116].

Let $s_{[j]}(A)$ denote the j th singular value of A . Let $\|A\|_k = s_{[1]}(A) + \dots + s_{[k]}(A)$ for $k = 1, 2, \dots, n$. Then (5) can be restated as $\|A - B\|_k \leq \|\lambda_{\downarrow}(A) - \lambda_{\uparrow}(B)\|_k$, $k = 1, 2, \dots, n$. So the second inequality in (1) holds for this special class of norms and hence, by a well-known theorem of Ky Fan, for every unitarily invariant norm. (See [4].)

It should be remarked that Sunder's paper contains a stronger result in that it also establishes an analogue of the second inequality in (1) for the case when A, B and $A - B$ are all normal. Under these conditions an analogue of the first inequality in (1) has been established in [1].

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