ON SEMIGROUPS GENERATED BY m -ACCRETIVE OPERATORS IN A STRICT SENSE

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ABSTRACT. Let $\{S(t): t \geq 0\}$ be a nonlinear semigroup generated by an m -accretive operator A in a real Banach space $(X, |\cdot|)$. It is shown that (1) for any $x \in \overline{D(A)}$, $S(\cdot)x$ belongs to $L^p(0, T; V)$ ($T > 0$) if A satisfies

$$|u_1 - u_2|^p + C\lambda \|u_1 - u_2\|^p \leq |u_1 + \lambda Au_1 - u_2 - \lambda Au_2|^p$$

($p > 1, C > 0$) for $\lambda > 0$ and $u_i \in D(A)$ ($i = 1, 2$), where $(V, \|\cdot\|)$ is a Banach space including $D(A)$ and included continuously in X ; and that (2) $A + B$ has similar properties to those of the above A if B is a Lipschitz continuous operator from V to X .

1. Introduction. Let $(X, |\cdot|)$ and $(V, \|\cdot\|)$ be real Banach spaces such that V is included in X algebraically and continuously. Consider an operator A in X (possibly nonlinear and multivalued) satisfying

$$(A) \quad V \supset D(A) \quad (\text{the domain of } A);$$

$$(B) \quad |u_1 - u_2|^p + C\lambda \|u_1 - u_2\|^p \leq |u_1 + \lambda Au_1 - u_2 - \lambda Au_2|^p$$

for all $\lambda > 0$ and $u_i \in D(A)$ ($i = 1, 2$), where $p > 1$ and $C > 0$ are constants;

$$(C) \quad X = R(I + \lambda A) \quad (\text{the range of } I + \lambda A) \text{ for all } \lambda > 0.$$

Clearly conditions (B) and (C) imply that the above A is m -accretive in X and hence generates a semigroup $\{S(t): t \geq 0\}$ on $\overline{D(A)}$ (the closure in X of $D(A)$). Crandall and Liggett [1] demonstrated that for each $x \in \overline{D(A)}$, $(I + (t/n)A)^{-n}x$ converges to $S(t)x$ in X as $n \rightarrow \infty$ uniformly on every bounded subinterval of $[0, \infty)$ and that $\{S(t): t \geq 0\}$ becomes a continuous semigroup of nonexpansive self-mappings of $\overline{D(A)}$.

The objective in this paper is to study some properties of $\{S(t): t \geq 0\}$ generated by an operator A satisfying the conditions (A)–(C). We will show below, among other things, that for any $x \in \overline{D(A)}$, $S(\cdot)x$ belongs to $L^p(0, T; V)$ for $T > 0$, and that if an operator $B: V \rightarrow X$ is Lipschitz continuous, then $A + B$ has similar properties to those of A .

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The condition (B) is, as is easily seen, equivalent to each of the following:

$$(1.1) \quad \tau(u_1 - u_2, Au_1 - Au_2)|u_1 - u_2|^{p-1} \geq (C/p)\|u_1 - u_2\|^p,$$

where $\tau(x, y) = \inf_{\lambda > 0} \lambda^{-1}(|x + \lambda y| - |x|)$ for $x, y \in X$;

$$(1.2) \quad |u_1 - u_2|^p + (C/p)\lambda\|u_1 - u_2\|^p \leq |u_1 + \lambda Au_1 - u_2 - \lambda Au_2||u_1 - u_2|^{p-1}.$$

REMARK. In the case that X is a Hilbert space, condition (1.1) with $p = 2$ is nothing but an inequality of Gårding type. It is also evident that the inequality

$$|u_1 - u_2| + K\lambda^\alpha\|u_1 - u_2\| \leq |u_1 + \lambda Au_1 - u_2 - \lambda Au_2|$$

for $\lambda > 0$, $u_i \in D(A)$ ($i = 1, 2$) and constants $K > 0$ and α with $0 < \alpha < 1$, implies (B) with $p = \alpha^{-1}$ and $C = K^p$. Therefore, condition (B) is considered to be rather general.

2. Smoothness of $S(t)$. By “smoothness” we mean that for a.a. $t > 0$, $S(t)\overline{D(A)}$ is included in V . The purpose of this section is to prove the following theorem.

THEOREM 1. *Let $\{S(t) : t \geq 0\}$ be the semigroup on $\overline{D(A)}$ generated by A satisfying (A)–(C). Then, the following statements hold.*

(i) *For each $x \in \overline{D(A)}$ and $T > 0$, $S(\cdot)x$ belongs to $L^p(0, T; V)$.*

(ii)

$$|S(t)x - S(t)y|^p + C \int_0^t \|S(r)x - S(r)y\|^p dr \leq |x - y|^p$$

for all $x, y \in \overline{D(A)}$ and $t \geq 0$.

(iii)

$$\begin{aligned} & |S(t)x - v|^p - |x - v|^p + C \int_0^t \|S(r)x - v\|^p dr \\ & \leq p \int_0^t \tau(S(r)x - v, -Av)|S(r)x - v|^{p-1} dr \end{aligned}$$

for all $x \in \overline{D(A)}$, $v \in D(A)$ and $t \geq 0$.

Before giving the proof, we first prove several lemmas, of which we will make later use, under the hypothesis of the theorem.

LEMMA 1. (i) *For each $x \in X$, $J_\lambda x = (I + \lambda A)^{-1}x$ is continuous in $\lambda \in (0, 1]$ in the topology of V .*

(ii) *For each $u \in D(A)$, $J_\lambda u$ is continuous in $\lambda \in [0, 1]$ in the topology of V , where $J_0 = I$.*

PROOF. From (B) we see that

$$(2.1) \quad \|J_\lambda x - J_\lambda y\| \leq (C\lambda)^{-1/p}|x - y|$$

for all $\lambda > 0$ and $x, y \in X$.

Let $x \in X$ and $0 < \mu < \lambda$. Then, using (2.1) and the formula

$$J_\lambda x = J_\mu(\mu\lambda^{-1}x + (\lambda - \mu)\lambda^{-1}J_\lambda x),$$

we have

$$\begin{aligned} \|J_\lambda x - J_\mu y\| &\leq (C\mu)^{-1/p} |\mu\lambda^{-1}x + (\lambda - \mu)\lambda^{-1}J_\lambda x - x| \\ &= (C\mu)^{-1/p} \cdot \lambda^{-1}(\lambda - \mu) |J_\lambda x - x|, \end{aligned}$$

which implies that (i) is true. For the proof of (ii) we have only to show the continuity at $\lambda = 0$ of $J_\lambda u$ for $u \in D(A)$. Again using (2.1), we obtain

$$\|J_\lambda u - u\| \leq (C\lambda)^{-1/p} |u - (I + \lambda A)u| = C^{-1/p} \cdot \lambda^{1-1/p} |Au|. \quad \text{Q.E.D.}$$

LEMMA 2. Let $\{S(t): t \geq 0\}$ be the semigroup generated in X by A . Then, for each $u \in D(A)$ and $t \geq 0$, $S(t)u$ belongs to V , and the following statements hold.

(i) $\|J_{t/n}^n u - S(t)u\| \rightarrow 0$ as $n \rightarrow \infty$ uniformly on every bounded subinterval of $[0, \infty)$.

(ii) $S(t)u$ is continuous in $t \geq 0$ in the topology of V and satisfies, for all $t, s \geq 0$,

$$\|S(t)u - S(s)u\| \leq 2(p/C)^{1/p} |Au| |t - s|^{1-1/p}.$$

PROOF. By (1.1) we have

$$\|J_\lambda^m u - J_\mu^n u\|^p \leq (p/C) (|AJ_\lambda^m u| + |AJ_\mu^n u|) |J_\lambda^m u - J_\mu^n u|^{p-1}.$$

Since $|AJ_\lambda^n u| \leq |Au|$, this estimate becomes

$$\|J_\lambda^m u - J_\mu^n u\| \leq (2p/C)^{1/p} |Au|^{1/p} |J_\lambda^m u - J_\mu^n u|^{1-1/p}.$$

Using the estimate due to Crandall and Liggett [1, (1.9)], we obtain

$$\begin{aligned} (2.2) \quad \|J_\lambda^m u - J_\mu^n u\| &\leq (2p/C)^{1/p} |Au| \left[\left\{ (m\lambda - n\mu)^2 + n\mu(\lambda - \mu) \right\}^{1/2} \right. \\ &\quad \left. + \left\{ (m\lambda - n\mu)^2 + m\lambda(\lambda - \mu) \right\}^{1/2} \right]^{1-1/p} \end{aligned}$$

if $0 < \mu \leq \lambda$ and $0 < m \leq n$ (integers).

Putting $\mu = t/n$ and $\lambda = t/m$ yields

$$\|J_{t/m}^m u - J_{t/n}^n u\| \leq 2(p/C)^{1/p} |Au| \left\{ t(1/m + 1/n)^{1/2} \right\}^{1-1/p},$$

which implies that $J_{t/n}^n u$ converges in V to some $v(t)$ uniformly on every bounded subinterval of $[0, \infty)$. But, since $V \subset X$ continuously, $v(t)$ coincides with $S(t)u$ for all $t \geq 0$.

The assertion (ii) is proved by taking the limit in (2.2) with $m = n$, $\mu = t/n$ and $\lambda = s/m$. Q.E.D.

LEMMA 3. For all $u, v \in D(A)$ and $t \geq 0$

$$(2.3) \quad |S(t)u - S(t)v|^p + C \int_0^t \|S(r)u - S(r)v\|^p dr \leq |u - v|^p.$$

PROOF. Using (B), we have

$$C\lambda \|J_\lambda^i x - J_\lambda^i y\|^p \leq |J_\lambda^{i-1} x - J_\lambda^{i-1} y|^p - |J_\lambda^i x - J_\lambda^i y|^p$$

for $x, y \in X$ and $i = 1, 2, \dots, n$.

Summing up the above for $i = 1, 2, \dots, n$, we obtain

$$|J_\lambda^n x - J_\lambda^n y|^p + C \sum_{i=1}^n \lambda \|J_\lambda^i x - J_\lambda^i y\|^p \leq |x - y|^p,$$

which implies, by Lemma 1, that for all $u, v \in D(A)$ and $\lambda > 0$

$$|J_\lambda^n u - J_\lambda^n v|^p + C \int_\lambda^{(n+1)\lambda} \|J_\lambda^{[r/\lambda]} u - J_\lambda^{[r/\lambda]} v\|^p dr \leq |u - v|^p.$$

Noting Lemma 2 and taking the limit as $\lambda \downarrow 0$ with $t \geq \lambda$ and $n = [t/\lambda]$, we obtain (2.3). Q.E.D.

LEMMA 4. For all $u, v \in D(A)$ and $t \geq 0$

$$(2.4) \quad \begin{aligned} & |S(t)u - v|^p - |u - v|^p + C \int_0^t \|S(r)u - v\|^p dr \\ & \leq p \int_0^t \tau(S(r)u - v, -Av) |S(r)u - v|^{p-1} dr. \end{aligned}$$

PROOF. Put $x_i = J_\lambda^i u$. Then, using (1.1) and recalling the properties of $\tau(\cdot, \cdot)$, we have

$$\begin{aligned} (C/p) \|x_i - v\|^p & \leq \tau(x_i - v, Ax_i - Av) |x_i - v|^{p-1} \\ & = \tau(x_i - v, \lambda^{-1}(x_{i-1} - x_i) - Av) |x_i - v|^{p-1} \\ & \leq -\lambda^{-1} |x_i - v|^p + \lambda^{-1} |x_{i-1} - v| |x_i - v|^{p-1} \\ & \quad + \tau(x_i - v, -Av) |x_i - v|^{p-1} \\ & \leq (p\lambda)^{-1} |x_{i-1} - v|^p - (p\lambda)^{-1} |x_i - v|^p \\ & \quad + \tau(x_i - v, -Av) |x_i - v|^{p-1}. \end{aligned}$$

Consequently we obtain

$$|x_i - v|^p - |x_{i-1} - v|^p + C\lambda \|x_i - v\|^p \leq p\lambda \cdot \tau(x_i - v, -Av) |x_i - v|^{p-1}.$$

Summing up the above for $i = 1, 2, \dots, n$, we have

$$\begin{aligned} & |x_n - v|^p - |u - v|^p + C \sum_{i=1}^n \lambda \|x_i - v\|^p \\ & \leq p \sum_{i=1}^n \lambda \tau(x_i - v, -Av) |x_i - v|^{p-1} \end{aligned}$$

and hence

$$\begin{aligned} & |J_\lambda^n u - v|^p - |u - v|^p + C \int_\lambda^{(n+1)\lambda} \|J_\lambda^{[r/\lambda]} u - v\|^p dr \\ & \leq p \int_\lambda^{(n+1)\lambda} \tau(J_\lambda^{[r/\lambda]} u - v, -Av) |J_\lambda^{[r/\lambda]} u - v|^{p-1} dr. \end{aligned}$$

Since $\tau(\cdot, \cdot)$ is upper semicontinuous and the integrand of the right is dominated by $p|Av|(|r|Au| + |u - v|)^{p-1}$, going to the limit as $\lambda \downarrow 0$ with $t \geq \lambda$ and $n = [t/\lambda]$, we obtain (2.4) (see [3, pp. 249–250]). Q.E.D.

PROOF OF THEOREM 1. Take $x \in \overline{D(A)}$ and let $\{u_n\}_{n=1}^\infty$ be a sequence in $D(A)$ such that $|u_n - x| \rightarrow 0$ as $n \rightarrow \infty$. Using (2.3), we have, for $T > 0$,

$$|S(T)u_m - S(T)u_n|^p + C \int_0^T \|S(r)u_m - S(r)u_n\|^p dr \leq |u_m - u_n|^p,$$

which implies that $S(\cdot)u_n$ converges in the topology of the Banach space $L^p(0, T; V)$ to some w as $n \rightarrow \infty$.

But, V is included in X continuously and $S(\cdot)u_n$ tends to $S(\cdot)x$ in $L^p(0, T; X)$ as $n \rightarrow \infty$. Hence, $w(t)$ must coincide with $S(t)x$ for a.a. $t \in (0, T)$, which completes the proof of (i).

The proof of (ii) and (iii) is simple. Let $x, y \in \overline{D(A)}$ and $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ be sequences in $D(A)$ such that $|u_n - x| \rightarrow 0$ and $|v_n - y| \rightarrow 0$ as $n \rightarrow \infty$. We have only to replace u, v in (2.3) and u in (2.4) by u_n, v_n and u_n respectively, and only to take the limit as $n \rightarrow \infty$. Q.E.D.

3. Perturbation for $S(t)$. Let A be an operator satisfying (A)–(C). The purpose of this section is to give conditions on an operator B in X under which $A + B$ also satisfies such conditions as (A)–(C).

We impose on B the following:

(D) $V \subset D(B);$

(E) There is a constant $L \geq 1$ such that $|Bu - Bv| \leq L\|u - v\|$
for all $u, v \in V$.

The Lipschitz continuity condition of $B: V \rightarrow X$ is simple but new. In fact, it is possible for B to stand for a “differential” operator. In perturbation theory in this direction for nonlinear semigroups, A was assumed to be m -accretive simply but B was sometimes forced to satisfy the condition $\overline{D(A)} \subset D(B)$ instead (see e.g. [4, 6]), which is too restrictive for B to become a “differential” operator.

Our method can be used to treat the same problem in the case that B satisfies a “local” Lipschitz continuity condition from V to X . In this paper, however, to clarify our idea we confine ourselves to conditions (D) and (E).

THEOREM 2. *If A satisfies (A)–(C) and B satisfies (D) and (E), then $A + B$ has the following properties:*

(a) $V \supset D(A + B);$

(b) $(1 - \omega\lambda)|u_1 - u_2|^p + (C/(2p))\lambda\|u_1 - u_2\|^p$
 $\leq |u_1 - \lambda(A + B)u_1 - u_2 - \lambda(A + B)u_2||u_1 - u_2|^{p-1}$

for all $\lambda > 0$ and $u_i \in D(A + B)$ ($i = 1, 2$);

(c) $R(I + \lambda(A + B)) = X$ for all λ with $0 < \lambda < \omega^{-1}$,

where $\omega = (1 - 1/p)L^{p/(p-1)}(2/C)^{1/(p-1)}$.

PROOF. Since $D(A + B) = D(A)$, (a) is obvious.

Using (1.2) and (E), we have for $\lambda > 0$ and $u_i \in D(A)$ ($i = 1, 2$)

$$(3.1) \quad \begin{aligned} & |u_1 - u_2|^p + (C/p)\lambda \|u_1 - u_2\|^p \\ & \leq (|u_1 + \lambda(A + B)u_1 - u_2 - \lambda(A + B)u_2| + L\lambda \|u_1 - u_2\|) |u_1 - u_2|^{p-1}. \end{aligned}$$

Applying to the right the inequality

$$(3.2) \quad L \cdot YZ^{p-1} \leq (C/(2p))Y^p + \omega Z^p \quad \text{for nonnegative } Y \text{ and } Z,$$

which is a simple consequence of Young's inequality, we see that (3.1) implies (b).

To prove (c) we will show that for each $x \in X$ there exists $u \in D(A)$ such that $u + \lambda(A + B)u = x$. Put $Tu = (I + \lambda A)^{-1}(x - \lambda Bu)$ for $\lambda > 0$. Then the operator T maps V into $D(A)$ and hence into V . In view of (1.2) we see that T satisfies

$$\begin{aligned} |Tu_1 - Tu_2|^p + (C/p)\lambda \|Tu_1 - Tu_2\|^p & \leq \lambda \|Bu_1 - Bu_2\| |Tu_1 - Tu_2|^{p-1} \\ & \leq L\lambda \|u_1 - u_2\| |Tu_1 - Tu_2|^{p-1} \end{aligned}$$

for all $\lambda > 0$ and $u_i \in V$ ($i = 1, 2$).

Again using (3.2), we have

$$(1 - \omega\lambda) |Tu_1 - Tu_2|^p + (C/p)\lambda \|Tu_1 - Tu_2\|^p \leq (C/(2p))\lambda \|u_1 - u_2\|^p.$$

Hence, if $0 < \lambda < \omega^{-1}$, $\|Tu_1 - Tu_2\| \leq 2^{-1/p} \|u_1 - u_2\|$. Since $2^{-1/p}$ is smaller than 1, a well-known fixed point theorem applies. Let u be the unique fixed point of T in V . Then $u = (I + \lambda A)^{-1}(x - \lambda Bu)$, which implies that u belongs to $D(A)$ and satisfies $u + \lambda(A + B)u = x$. Q.E.D.

EXAMPLE. Let Ω be a bounded domain in R^N with smooth boundary. Consider the natural realization A in $L^2(\Omega)$ of $-\Delta$, $\Delta = \sum_{i=1}^N (\partial/\partial x_i)^2$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, where $H^k(\Omega)$ and $H_0^k(\Omega)$ denote the usual Sobolev spaces—the real Hilbert spaces with norm $\|\cdot\|_k$ and inner product $(\cdot, \cdot)_k$. Integration by parts yields $(Au, u)_0 = \|u\|_1^2 - \|u\|_0^2$ for $u \in D(A)$, a special form of Gårding's inequality. Hence, the linear operator A satisfies the condition

$$(1 - \lambda) \|u\|_0^2 + \lambda \|u\|_1^2 \leq \|u + \lambda Au\|_0 \|u\|_0$$

as well as $R(I + \lambda A) = L^2(\Omega)$ for $0 < \lambda < 1$ (Lax-Milgram Lemma). Next, the operator B defined on $H_0^1(\Omega)$ by $Bu = (1 + |\text{grad } u|^2)^{1/2}$ clearly satisfies

$$\|Bu - Bv\|_0 \leq \|u - v\|_1 \quad \text{for } u, v \in H_0^1(\Omega).$$

Thus, we can conclude by Theorems 1 and 2 that for the nonlinear semigroup $\{T(t): t \geq 0\}$ in $L^2(\Omega)$ generated by the semilinear operator $A + B$, $T(t)L^2(\Omega)$ is included in $H_0^1(\Omega)$ for a.a. $t > 0$.

REMARK. It is known [2, 5] that if X is a “nice” Banach space, then there is a biunique correspondence between m -accretive operators in X and nonlinear semigroups on nonexpansive retracts of X . It would be of interest to determine if there is an analogous correspondence between m -accretive operators satisfying (B) and nonlinear semigroups which satisfy (ii) of Theorem 1.

The author wishes to thank the referee for useful suggestions.

NOTE ADDED IN PROOF. Since this paper was submitted for publication, we have become aware that the theory in §2 resembles that in §3 of the paper:

A. Pazy, *The Lyapunov method for semigroups of nonlinear contractions in Banach spaces*, J. Analyse Math. **40** (1981), 239–262.

Here, the semigroup generated by A which satisfies instead of (B)

$$\phi(u) + \lambda\psi(u) \leq \phi(u + \lambda Au) \quad \text{for } u \in D(A) \text{ and } \lambda > 0,$$

where ϕ and ψ are lower semicontinuous functions, roughly speaking, is discussed among other things.

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