ON THE DIEUDONNÉ PROPERTY FOR $C(\Omega, E)$

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ABSTRACT. In a recent paper, F. Bombal and P. Cembranos showed that if E is a Banach space such that E^* is separable, then $C(\Omega, E)$, the Banach space of continuous functions from a compact Hausdorff space Ω to E , has the Dieudonné property. They asked whether or not the result is still true if one only assumes that E does not contain a copy of l_1 . In this paper we give a positive answer to their question. As a corollary we show that if E is a subspace of an order continuous Banach lattice, then E has the Dieudonné property if and only if $C(\Omega, E)$ has the same property.

If E is a Banach space and Ω is a compact Hausdorff space, then $C(\Omega, E)$ will stand for the Banach space of the E -valued continuous functions on Ω under the supremum norm.

A Banach space E is said to have the *Dieudonné property* if for every Banach space F , any bounded linear operator $T: E \rightarrow F$ that transforms weakly Cauchy sequences into weakly convergent sequences is weakly compact. In [3] F. Bombal and P. Cembranos showed that if E is a Banach space such that E^* is separable, then $C(\Omega, E)$ has the Dieudonné property and they asked whether the same result is true when replacing the assumption that E^* is separable by supposing only that l_1 does not embed in E . In this paper we give a positive answer to their question.

Recall that a topological space (X, γ) is said to be Polish if it is homeomorphic to a separable complete metric space and it is said to be analytic if it is the continuous image of a Polish space. A subset A of a topological space (X, γ) is said to be coanalytic if its complement $(X \setminus A, \gamma)$ is analytic. Finally A is said to be PCA if it is the continuous image of a coanalytic space.

The notations and terminology used and not defined can be found in [5, 8, or 10]. In the proof of Lemma 3 we need the following two results.

THEOREM 1 (M. SREBRNY [9]). *Let X and Y be two analytic spaces and let F be a multivalued function from X to the subsets of Y , such that its graph is PCA and for which one can prove that for every $x \in X$, $F(x) \neq \emptyset$ using only the axioms of ZFC. Then there exists a universally measurable map $f: X \rightarrow Y$ such that $f(x) \in F(x)$ for every $x \in X$.*

THEOREM 2 (I. ASSANI [1, 2]). *Let E be a separable Banach space. The set of weakly Cauchy sequences is a coanalytic subset of $E^{\mathbb{N}}$.*

Received by the editors January 5, 1985.

1980 *Mathematics Subject Classification*. Primary 46G10, 46B22.

Key words and phrases. Dieudonné property.

¹Supported by NSF grant DMS-8301099.

² The second and third authors were supported by a Summer Research Fellowship from the University of Missouri.

LEMMA 3. *Let E be a separable Banach space containing no isomorphic copy of l_1 . Then there is a sequence of maps $\theta_n: E^* \rightarrow E$ so that each θ_n is universally measurable (for the weak*-topology of E^*) and, for each $e^* \in E^*$, we have*

- (i) $(\theta_n(e^*))$ is a weak Cauchy sequence,
- (ii) $\|\theta_n(e^*)\| \leq 1$ for $n \geq 1$,
- (iii) $\lim_{n \rightarrow \infty} \langle \theta_n(e^*), h \rangle = \langle e^*, h \rangle$.

PROOF. Let B be the closed unit ball of E and let H be the subset of $B^{\mathbb{N}}$ consisting of those sequences in E that are weakly Cauchy. By Theorem 2, H is a coanalytic subset of $B^{\mathbb{N}}$. Let F be the multivalued function from E^* to the subsets of $B^{\mathbb{N}}$ defined by

$$F(e^*) = \left\{ (e_n) \in B^{\mathbb{N}}; (e_n) \text{ is weak Cauchy and } \lim_{n \rightarrow \infty} \langle e_n, e^* \rangle = \|e^*\| \right\}.$$

Rosenthal's Theorem [7] guarantees that $F(e^*) \neq \emptyset$ for every $e^* \in E^*$. We endow E^* with its weak*-topology and the graph $G(F)$ of F is the set $\{(e^*, (e_n)); (e_n) \in F(e^*)\}$ which is a subset of $E^* \times H$.

Consider the map $\psi: E^* \times H \rightarrow \mathbf{R}$ defined by

$$\psi(e^*, (e_n)) = \lim_{n \rightarrow \infty} \langle e_n, e^* \rangle - \|e^*\|.$$

It is easy to see that ψ is a Borel map and $G(F) = \psi^{-1}(0)$. Hence $G(F)$ is a Borel subset of $E^* \times H$ and hence it is coanalytic in $E^* \times B^{\mathbb{N}}$.

By Theorem 1, there exists a universally measurable map $f: E^* \rightarrow E^{\mathbb{N}}$ such that $f(e^*) \in F(e^*)$ for every $e^* \in E^*$. For each $n \geq 1$, let $\pi_n: E^{\mathbb{N}} \rightarrow E$ be the n th projection $\pi_n((e_k)) = e_n$. To finish the proof, it is enough to define θ_n by $\theta_n(e^*) = \pi_n(f(e^*))$.

The proof of the following theorem follows the steps of Theorem 3 in [4]. The following sketch is given for the reader's convenience.

THEOREM 4. *Let E be a Banach space containing no isomorphic copy of l_1 , and let Ω be a compact Hausdorff space. Then $C(\Omega, E)$ has the Dieudonné property.*

PROOF. It is enough to prove the theorem for E separable and Ω metrizable. Let $T: C(\Omega, E) \rightarrow X$ be a bounded linear operator that takes weakly Cauchy sequences into weakly convergent sequences.

Let V be the unit ball of $C(\Omega, E)$ and let $W = \overline{T(V)}$ in X . We will show that W is weakly compact by proving that every $x^* \in X^*$ attains its maximum on W (James [6]).

Fix $x^* \in X^*$, with $\|x^*\| \leq 1$. Then $\sup_{w \in W} x^*(w) = \|T^*x^*\|$.

As in [4], let G be the representing measure of T , let λ be its control measure and let $h: \Omega \rightarrow E^*$ be a weak*-Borel λ -measurable map from $\Omega \rightarrow E^*$ such that for each $f \in C(\Omega, E)$, $\langle f, T^*x^* \rangle = \int_{\Omega} \langle f(\omega), h(\omega) \rangle d\lambda(\omega)$ and $\|T^*x^*\| = \int_{\Omega} \|h(\omega)\| d\lambda$.

Now for each $\omega \in \Omega$, let $\psi_n(\omega) = \theta_n(h(\omega))$, where θ_n are the maps obtained in Lemma 3. Each $\psi_n: \Omega \rightarrow E$ is universally measurable and

- (i) $\psi_n(\omega)$ is weak-Cauchy in E for every $\omega \in \Omega$,
- (ii) $\|\psi_n(\omega)\| \leq 1$ for $n \geq 1$ and $\omega \in \Omega$,
- (iii) $\lim_{n \rightarrow \infty} \langle \psi_n(\omega), h(\omega) \rangle = \|h(\omega)\|$.

Let $\varepsilon_n > 0$ be any decreasing sequence so that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Choose $\delta_n > 0$ a decreasing sequence so that if $\lambda(B) < \delta_n$, then $\|G\|(B) < \varepsilon_n/2$. For each $n \geq 1$ let

Ω_n be a compact subset of Ω so that $\lambda(\Omega_n) > 1 - \delta_n$ and each ψ_k is continuous on Ω_n . Of course we can assume that Ω_n is an increasing sequence. By the Borsuk-Dugundji theorem [10] there is an extension operator $S_n: C(\Omega_n, E) \rightarrow C(\Omega, E)$ so that $\|S_n\| = 1$ and $S_n(f(\omega)) = f(\omega)$ for $f \in C(\Omega_n, E)$ and $\omega \in \Omega_n$. Let $g_{n,k} = S_n(\psi_k|_{\Omega_n})$.

By construction of the ψ_k , the sequence $(\psi_k(\omega))$ is a weakly Cauchy sequence in E for each $\omega \in \Omega$. This shows that $(\psi_k|_{\Omega_n})$ is a weak Cauchy sequence in $C(\Omega_n, E)$, hence $(g_{n,k})_{k \geq 1}$ is a weakly Cauchy sequence in $C(\Omega, E)$. Therefore $x_{n,k} = T(g_{n,k})$ is weakly convergent in X and $x_{n,k} \in W$. Let $w_n = \text{weak-lim}_k x_{n,k}$ for each $n \geq 1$. Hence $w_n \in W$ for each $n \geq 1$.

It follows as in [4] that $w = \lim_n w_n$ exists, belongs to W and

$$x^*(w) = \int_{\Omega} \|h(\omega)\| d\lambda = \sup_{u \in W} x^*(u).$$

Hence x^* attains its maximum on W at w .

COROLLARY 5. *Let E be a subspace of an order continuous Banach lattice. Then E has the Dieudonné property if and only if $C(\Omega, E)$ has the Dieudonné property.*

PROOF. If E contains a copy of l_1 , then E would contain a complemented copy of l_1 [4, 11], but this is impossible if E has the Dieudonné property. Apply Theorem 4 to finish the proof.

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