GLOBAL STABILITY OF A BIOLOGICAL MODEL WITH TIME DELAY

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ABSTRACT. This paper gives necessary and sufficient conditions for global stability of certain logistic delay differential equations for all values of the delay.

Biological models frequently lead to delay differential equations and to questions concerning the stability of equilibrium solutions of such models. The monographs by Cushing [6] and MacDonald [16] discuss a number of examples of such models from population dynamics, ecology, and physiology. Much of the work with such models has focused on delay equations which are reducible, through a change of variables, to systems of differential equations without delays [3, 4, 8, 14, 18, 19].

Another line of research is concerned with conditions under which linear retarded functional differential equations of the form

$$\dot{x}(t) = ax(t) + \sum_{i=1}^L b_i x(t - r_i), \quad r_i \geq 0,$$

are asymptotically stable for all values of the delay [1, 3, 5, 7, 9, 11, 12, 14]. We propose to combine these lines of research and examine global stability of the widely used logistic model of population dynamics,

$$(1) \quad \dot{x}(t) = x(t) \left(r + ax(t) + \sum_{i=1}^L b_i x(t - r_i) \right), \quad r_i \geq 0,$$

$$x(t) = \phi(t) \quad \text{for } -r \leq t \leq 0,$$

where $r = \max\{r_i\}$.

We establish global stability (with respect to solutions with positive initial conditions) of the positive equilibrium $\bar{x}$ of equation (1). The hypothesis of our theorem guarantees that such an equilibrium exists. Koslesov [13] and Mazorov [15] point out that equation (1) can possess periodic solutions, while Beddington and May [2] point out that it need not do so. Hence the question of global stability of equation (1) is of interest.

THEOREM. *Equation (1) is globally stable for initial conditions ϕ satisfying $\phi(0) \geq 0$ and for all $r_i \geq 0$ if and only if*

$$(2) \quad \sum_{i=1}^L |b_i| \leq -a$$

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and

$$(3) \quad \sum_{i=1}^L b_i < -a.$$

PROOF. First we assume conditions (2) and (3) hold. Let $\bar{x}$ be a positive equilibrium for equation (1) and let $y = x - \bar{x}$. Then

$$\frac{dy}{dt} = (\bar{x} + y(t)) \left(ay(t) + \sum_{i=1}^L b_i y(t - r_i) \right).$$

For each nonzero b_i , let d_i be a positive constant to be chosen later. Define the positive definite functional $V: C(-r, 0) \rightarrow \mathbf{R}$ by

$$V(\phi) = \phi(0) - \bar{x} \ln \left(\frac{\phi(0) + \bar{x}}{\bar{x}} \right) + \sum_{i=1}^L d_i \int_{-r_i}^0 \phi^2(\theta) d\theta.$$

We will prove that V is a Liapunov function for equation (1). Using notation from Hale [10],

$$\begin{aligned} \dot{V}(\phi) &= a\phi^2(0) + \sum_{i=1}^L b_i \phi(-r_i) \phi(0) + \phi^2(0) \sum_{i=1}^L d_i - \sum_{i=1}^L d_i \phi^2(-r_i) \\ &\leq a\phi^2(0) + \sum_{i=1}^L |b_i| |\phi(-r_i)| |\phi(0)| + \phi^2(0) \sum_{i=1}^L d_i - \sum_{i=1}^L d_i \phi^2(-r_i) \\ &= -z^T A z \end{aligned}$$

where $z^T = (|\phi(0)|, |\phi(-r_1)|, |\phi(-r_2)|, \dots, |\phi(-r_L)|)$ and A is the $(L+1)$ square matrix

$$A = \begin{bmatrix} -a - \sum_{i=1}^L d_i & -|b_1|/2 & -|b_2|/2 & \cdots & -|b_L|/2 \\ -|b_1|/2 & d_1 & 0 & \cdots & 0 \\ -|b_2|/2 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|b_L|/2 & 0 & 0 & \cdots & d_L \end{bmatrix}$$

We will have

$$-z^T A z \leq -\lambda_{\min} |\phi(0)|^2 < 0,$$

where $\lambda_{\min}$ is the smallest eigenvalue of A , if matrix A is positive definite. Since the off-diagonal elements are nonpositive, A is positive definite if and only if all the principal minors of A are positive [17]. This condition implies that the constants $d_1, \dots, d_L$ must be chosen so that

$$(4) \quad \sum_{i=1}^L \frac{b_i^2}{4d_i} + \sum_{i=1}^L d_i < -a.$$

First, consider the case where

$$(5) \quad \sum_{i=1}^L |b_i| < -a,$$

and choose $d_i = |b_i|/2$. Inequality (4) holds for this choice of d_i 's. Thus, equation (1) is globally stable when (5) holds.

Consider the case when

$$(6) \quad \sum_{i=1}^L |b_i| = -a \quad \text{and} \quad \sum_{i=1}^L b_i < -a.$$

These two conditions imply that some b_i is negative. For the same choice of d_i , note that

$$(7) \quad a = -\sum_{i=1}^L |b_i| = -2 \sum_{i=1}^L d_i$$

and that we can now only show that $\dot{V}(\phi) \leq 0$.

To circumvent this problem, consider the set $S = \{\phi: \dot{V}(\phi) = 0\}$.

Using (7) and completing the square,

$$\begin{aligned} \dot{V}(\phi) &= -\left[\left(-a - \sum_{i=1}^L d_i \right) \phi^2(0) + \sum_{i=1}^L \left(d_i \phi^2(-r_i) - b_i \phi(0) \phi(-r_i) \right) \right] \\ &= -\left[\left(-a - \sum_{i=1}^L d_i - \sum_{i=1}^L \frac{b_i^2}{4d_i} \right) \phi^2(0) \right. \\ &\quad \left. + \sum_{i=1}^L d_i \left(\phi^2(-r_i) - \frac{b_i}{d_i} \phi(0) \phi(r_i) + \left(\frac{b_i}{2d_i} \right)^2 \phi^2(0) \right) \right] \\ &= -\left[\left(-a - \sum_{i=1}^L \left(d_i + \frac{b_i^2}{4d_i} \right) \right) \phi^2(0) + \sum_{i=1}^L d_i \left(\phi(-r_i) - \frac{b_i}{2d_i} \phi(0) \right)^2 \right] = 0. \end{aligned}$$

Since $-a - \sum_{i=1}^L (d_i + b_i^2/4d_i) = 0$,

$$S = \{\phi: \phi(-r_i) = (b_i/2d_i)\phi(0) \text{ for all } i = 1, \dots, L\}.$$

Now determine the largest set M in S which is invariant with respect to equation (1). M must be the set of initial values of solutions satisfying

$$(8) \quad y(t - r_i) = (b_i/2d_i)y(t) \quad \text{for all } i = 1, \dots, L.$$

This implies

$$\begin{aligned} \frac{dy(t)}{dt} &= (\bar{x} + y(t)) \left(ay(t) + \sum_{i=1}^L b_i y(t - r_i) \right) \\ &= (\bar{x} + y(t)) \left(a + \sum_{i=1}^L \frac{b_i^2}{2d_i} \right) y(t) = 0 \quad \text{by (7)}. \end{aligned}$$

Thus $y(t) = c$, a constant. But (8) gives $c = (b_i/2d_i)c$ for all $i = 1, \dots, L$. Since some b_i is negative, $b_i/2d_i \neq 1$ for some i , and we must have $c = 0$. By Theorem 3.1 of Hale [10], equation (1) is globally asymptotically stable.

To prove the converse, let $y = x - \bar{x}$ and linearize the transformed equation about $y = 0$ to obtain

$$(9) \quad \frac{dy}{dt} = \bar{x} \left(ay(t) + \sum_{i=1}^L b_i y(t - r_i) \right).$$

From Corollary 1.2 of Lenhart and Travis [14], equation (9) is stable if and only if conditions (2) and (3) hold. Since global stability of the nonlinear equation (1) implies stability of the linearized equation (9), conditions (2) and (3) must hold.

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