SOME WEAKLY COMPACT OPERATORS BETWEEN BANACH LATTICES DO NOT FACTOR THROUGH REFLEXIVE BANACH LATTICES

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ABSTRACT. We construct an example proving the claim of the title.

1. Introduction. A celebrated result of W. Davis, T. Figiel, W. Johnson and A. Pelczynski [2] states that a weakly compact operator between Banach spaces factors through a reflexive Banach space. It is hence natural to ask [1, problems 1, 2; 2, problems 3, 4] if a weakly compact operator between Banach lattices factors through a reflexive Banach lattice. The answer is known to be positive under some extra assumptions. We shall prove that it is negative in general. More precisely, we have

THEOREM A. *There exists a weakly compact subset K of the positive cone of $C([0, 1])$ such that whenever E is a reflexive Banach lattice, and $T: E \rightarrow C([0, 1])$ is a bounded operator, $T(E_1)$ does not contain K (where $E_1 = \{x \in E; \|x\| \leq 1\}$).*

Let (x_n) be a dense sequence in K . Let $U: l^1 \rightarrow C([0, 1])$ be such that $U(e_n) = x_n$, where (e_n) denotes the canonical basis of l^1 . Then U is positive weakly compact. Suppose, however, that one could factor U through a reflexive Banach lattice E : that is, $U = T \circ V$ for $V: l^1 \rightarrow E$ and $T: E \rightarrow C([0, 1])$. Since one can suppose $\|V\| \leq 1$, this shows that $K \subset T(E_1)$, a contradiction.

The compact we shall construct is homeomorphic to ω^{ω^2} .

Problem. What is the smallest ordinal α such that there exists a weak compact set $K \subset C([0, 1])$ homeomorphic to α that satisfies the conditions of the theorem?

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2. Line of attack. A reflexive Banach lattice will always be represented as a space of measurable functions on a measure space (Ω, Σ, μ) such that $L^\infty(\mu) \subset E \subset L^1(\mu)$ and $\|f\|_1 \leq \|f\|_E$ for $f \in E$ [3]. Then E^* is a space of measurable functions, and both the unit ball of E and E^* are equi-integrable.

The following technical fact is the cornerstone of the proof. It will be established in §3. We fix once and for all an ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

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MAIN FACT. Let E be a reflexive Banach lattice. Suppose that we are given $b > 0$, $\psi \in E_1^*$; for $n \in \mathbb{N}$, an element $\psi_n \in E_1^*$; $\phi \in E$; for $q \in \mathbb{N}$, an element ϕ_q of E ; and for $p \leq q$ and each sequence $p \leq i_1 < i_2 < \dots < i_p$ of $\mathbb{N}$, an element $\phi_q(i_1, \dots, i_p) \in E$ such that the following conditions hold:

$$(1) \psi = \text{weak-lim}_{n \rightarrow \mathcal{Q}} \psi_n.$$

$$(2) \text{ For each } p < q, \text{ each } p \leq i_1 < \dots < i_p,$$

$$\phi_q(i_1, \dots, i_p) = \text{weak-lim}_{i_{p+1} \rightarrow \mathcal{Q}} \phi_q(i_1, \dots, i_p, i_{p+1})$$

(for $p = 0$ this condition means $\phi_q = \text{weak-lim}_{i \rightarrow \mathcal{Q}} \phi_q(i)$).

$$(3) \phi = \text{weak-lim}_{q \rightarrow \mathcal{Q}} \phi_q.$$

$$(4) \text{ There exists } A > 0 \text{ with } \|\phi_q(i_1, \dots, i_p)\| \leq A \text{ whenever } p \leq q, p \leq i_1 < \dots < i_p.$$

$$(5) \text{ For each } p \leq q, p \leq i_1 < \dots < i_p < n,$$

$$\left| \int \psi_n(\phi_q(i_1, \dots, i_p, n) - \phi_q(i_1, \dots, i_p)) d\mu \right| \geq 1.$$

$$(6) \int |\psi \phi| d\mu > b.$$

Then there exist $p \leq q$, $p \leq i_1 < \dots < i_p$, and n such that

$$\int |\psi_n \phi_q(i_1, \dots, i_p)| d\mu > b + 1.$$

We now show how to derive the theorem from the fact. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let

$$L = \{h: \bar{\mathbb{N}} \rightarrow \{0, 1\}; \exists p \leq i_1 < \dots < i_p; h(i) = 0 \text{ for } i \neq i_1, \dots, i_p\}.$$

Then $L \subset C(\bar{\mathbb{N}})$ is weakly compact. Let $l > 0$. Consider the map θ_l from L' to $C(\bar{\mathbb{N}}')$, given by $\theta_l(h_1, \dots, h_l)(n_1, \dots, n_l) = 1$ if the number of indexes i for which $h_i(n_i) = 1$ is even, and $\theta_l(h_1, \dots, h_l)(n_1, \dots, n_l) = 0$ otherwise. The map θ_l is continuous when $C(\bar{\mathbb{N}}')$ is equipped with the pointwise convergence topology, so $K_l = \theta_l(L')$ is a weakly compact set.

For $t \in \bar{\mathbb{N}}^l$, we denote by δ_t the Dirac measure at t .

PROPOSITION 1. Let E be a reflexive Banach lattice, and $T: E \rightarrow C(\bar{\mathbb{N}}^l)$ an operator with $\|T\| \leq 1$. Then either $K_l \not\subset T(E)$ or

$$\sup_{x \in K_l} \inf_{\substack{f \in E \\ T(f) = x}} \sup_{t \in \bar{\mathbb{N}}^l} \int |T^*(\delta_t)f| d\mu \geq l.$$

PROOF. The proof is by induction over l . Assume that the result holds for $l - 1$ but not for l (the case $l = 1$ is identical to the general case). We can assume $K_l \subset T(E)$. There is $b < l$ such that for each $(h_1, \dots, h_l) \in L'$ there is $f(h_1, \dots, h_l)$ in E with

$$(7) \quad T(f(h_1, \dots, h_l)) = \theta_l(h_1, \dots, h_l)$$

and

$$(8) \quad \forall t \in \bar{\mathbb{N}}^l, \quad \int |T^*(\delta_t)f(h_1, \dots, h_l)| d\mu < b.$$

For $p < i_1 < \dots < i_p$ we have $1_{\{i_1, \dots, i_p\}} \in L$. Fix $q \in \mathbb{N}$. For $p \leq q$, $p < i_1 < \dots < i_p$, we define $g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p)$ in the following way. First we set

$$(9) \quad g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_q) = f(h_1, \dots, h_{l-1}, 1_{\{i_1, \dots, i_q\}}).$$

For $0 < p < q$ we define $g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p)$ by decreasing induction over p :

$$(10) \quad g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p) = \text{weak-lim}_{i_{p+1} \rightarrow \mathcal{U}} g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_{p+1}).$$

For $p = 0$ we set

$$(11) \quad g_q(h_1, \dots, h_{l-1}) = \text{weak-lim}_{i \rightarrow \mathcal{U}} g_q(h_1, \dots, h_{l-1}; i).$$

Finally, we set

$$(12) \quad g(h_1, \dots, h_{l-1}) = \text{weak-lim}_{q \rightarrow \mathcal{U}} g_q(h_1, \dots, h_{l-1}).$$

We identify $\bar{\mathbf{N}}^{l-1}$ to $\bar{\mathbf{N}}^{l-1} \times \{\infty\} \subset \bar{\mathbf{N}}^l$. Let P be the corresponding restriction map from $C(\bar{\mathbf{N}}^l)$ to $C(\bar{\mathbf{N}}^{l-1})$. We have $\|P \circ T\| \leq 1$. From (7) we see that

$$P \circ T(f(h_1, \dots, h_l)) = P(\theta_l(h_1, \dots, h_l)) = \theta_{l-1}(h_1, \dots, h_{l-1}).$$

Since T is weakly continuous, it follows by (9)–(12) that

$$P \circ T(g(h_1, \dots, h_{l-1})) = \theta_{l-1}(h_1, \dots, h_{l-1}).$$

Since $b - 1 < l - 1$, the induction hypothesis gives $(h_1, \dots, h_{l-1}) \in L^{l-1}$ and $t \in \bar{\mathbf{N}}^{l-1}$ such that

$$(13) \quad \int |(P \circ T)^*(\delta_t)g(h_1, \dots, h_{l-1})| d\mu > b - 1.$$

If $t = (n_1, \dots, n_{l-1})$, $t' = (n_1, \dots, n_{l-1}, \infty)$, this means that

$$\int |T^*(\delta_{t'})g(h_1, \dots, h_{l-1})| d\mu > b - 1.$$

Let $\psi = T^*(\delta_{t'})$. Since T^* is weakly continuous, we have

$$(14) \quad \psi = \text{weak-lim}_n \psi_n,$$

where $\psi_n = T^*(\delta_{(n_1, \dots, n_{l-1}, n)})$. Let $\phi = g(h_1, \dots, h_{l-1})$, $\phi_q = g_q(h_1, \dots, h_{l-1})$, and for $p \leq q$, $p \leq i_1 < \dots < i_p$, let

$$\phi_q(i_1, \dots, i_p) = g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p).$$

Then condition (1) follows from (14). Condition (2) follows from (10) and (11). Condition (3) follows from (13), condition (4) holds with $A = b$. Condition (6) follows from (13), and condition (5) follows from the definition of θ_l . The hypotheses of the main fact are satisfied. So there exists n, q , and $p \leq q$, $p < i_1 < \dots < i_p$, such that

$$\int |\psi_n \phi_q(i_1, \dots, i_p)| d\mu > b.$$

Condition (15) means that

$$\int |T^*(\delta_{(n_1, \dots, n_{l-1}, n)})g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p)| d\mu > b.$$

Let

$$\xi = \text{sign } T^*(\delta_{(n_1, \dots, n_{l-1}, n)}) g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p),$$

so

$$\int \xi T^*(\delta_{(n_1, \dots, n_{l-1}, n)}) g_q(h_1, \dots, h_{l-1}; i_1, \dots, i_p) d\mu > b.$$

Conditions (9) and (11) show that one can pick (by induction) $i_p < i_{p+1} < \dots < i_q$ such that

$$\int \xi T^*(\delta_{(n_1, \dots, n)}) f(h_1, \dots, h_{l-1}, 1_{\{i_1, \dots, i_q\}}) d\mu > b.$$

This contradicts (8) and finishes the proof.

We now conclude the proof of Theorem A. Denote by M the Alexandroff compactification of the discrete sum of the sets $\bar{N}^l$. Consider each K_l as a subset of $C(M)$ by extending the functions of K_l to zero outside $\bar{N}^l$. Let $K = \bigcup K_l$. By construction, K consists of functions taking the values zero or one, so K is contained in the positive cone of $C(M)$. Proposition 1 shows that if T is an operator from E to $C(M)$, then for $l > \|T\|$ we have $K_l \not\subset T(E_1)$, so $K \not\subset T(E_1)$. To conclude, one notices that there is a positive embedding of $C(M)$ in $C([0, 1])$, such that its image is complemented.

3. Proof of the Main Fact. Let $\varepsilon = \frac{1}{8}(\int |\phi\psi| d\mu - b)$. Let $b' = b + 6\varepsilon$, so we have $\int |\phi\psi| d\mu > b' + \varepsilon$. Let $\xi = \text{sign } \phi\psi$.

LEMMA 1. $\exists a > 0, \forall r, \exists q \geq r, \exists I \subset \mathbb{N}, I \in \mathcal{U}, \forall n \in I,$

$$(15) \quad \int_{\{|\psi_n| > a\}} |\psi_n \phi_q| d\mu \leq \varepsilon; \quad \int \xi \psi \phi_q d\mu > b'.$$

PROOF. Let

$$J = \left\{ q \in \mathbb{N}; \int \xi \psi \phi_q d\mu > b' \right\}.$$

Then J is infinite since $\int \xi \psi \phi d\mu > b'$ and since ϕ is a weak cluster point of $\{\phi_q; q \in J\}$. Let ϕ' be a weak cluster point of $\{\phi_q; q \in J\}$. Let $d > 0$ be large enough that $\|\phi' - \phi \wedge d\|_E \leq \varepsilon/3$. Since E_1^* is equi-integrable, there exists $a > 0$ large enough that

$$y \in E_1^* \Rightarrow \int_{\{|y| \geq a\}} |y| d\mu \leq \frac{\varepsilon}{3d}.$$

Since ϕ' is a weak cluster point of $\{|\phi_q|; q \in J; q \geq r\}$, there exists a finite subset J' of J , such that $q \geq r$ for $q \in J'$, and numbers $\alpha_q \geq 0$, $\sum_{q \in J'} \alpha_q = 1$, with $\|\sum \alpha_q |\phi_q| - \phi'\|_E \leq \varepsilon/3$.

For each n we get, with $H = \{|\psi_n| \geq a\}$,

$$\begin{aligned} \int_H \sum_{q \in J'} \alpha_q |\phi_q| |\psi_n| d\mu &= \int_H (\sum \alpha_q |\phi_q| - \phi') |\psi_n| d\mu \\ &\quad + \int_H (\phi' - \phi' \wedge d) |\psi_n| d\mu + \int_H \phi' \wedge d |\psi_n| d\mu \\ &\leq \|\sum \alpha_q |\phi_q| - \phi'\|_E \|\psi_n\|_{E*} + \|\phi' - \phi' \wedge d\|_E \|\psi_n\|_{E*} \\ &\quad + \|\phi' \wedge d\|_\infty \|\psi_n 1_H\|_1 \leq \varepsilon/3 + \varepsilon/3 + d\varepsilon/3d \leq \varepsilon. \end{aligned}$$

So for each n there is $q \in J'$ with $\int_H |\phi_q \psi_n| d\mu \leq \varepsilon$.

It follows that there is $q \in J'$ such that the set $I = \{n; \int |\phi_q \psi_n| d\mu \leq \varepsilon\}$ belongs to $\mathcal{U}$. The proof is complete.

Recall that E is represented as a space of measurable functions on (Ω, Σ, μ) . For $q \leq i_1 < i_2 < \dots < i_q$, let $\nu_q(i_1, \dots, i_q)$ be the distribution on $\mathbf{R}^{q+1}$ of the function $\Omega \rightarrow \mathbf{R}^{q+1}$ given by

$$(16) \quad \omega \rightarrow (\phi_q(\omega), \phi_q(i_1)(\omega), \dots, \phi_q(i_1, \dots, i_q)(\omega)).$$

If $E_A = \{f \in E, \|f\| \leq A\}$, let

$$a_n = \sup_{E_A} \int_{\{|f| \geq n\}} |f| d\mu.$$

Since E_A is equi-integrable, we have $a_n \rightarrow 0$. Denote $x \in \mathbf{R}^{q+1}$ by $(x_1, \dots, x_{q+1})$. Let L_q be the set of probabilities ν on $\mathbf{R}^{q+1}$ that satisfy the condition

$$\forall n, \forall i \leq q+1, \quad \int_{\{|x_i| \geq n\}} |x_i| d\nu(x) \leq a_n.$$

Then L_q is weakly compact, and $\nu_q(i_1, \dots, i_q) \in L_q$ for each $q < i_1 < \dots < i_q$. Hence, we can define

$$(17) \quad \nu_q = \lim_{i_1 \rightarrow \mathcal{U}} \lim_{i_2 \rightarrow \mathcal{U}} \dots \lim_{i_q \rightarrow \mathcal{U}} \nu_p(i_1, \dots, i_q),$$

where the limit is in the weak sense. We now fix q . For $i \leq q+1$ we denote by Σ_i the σ -algebra on $\mathbf{R}^{q+1}$ generated by the coordinates of rank $\leq i$.

LEMMA 2. *When $\mathbf{R}^{q+1}$ is equipped with ν_q , the sequence $(x_i)_{i \leq q+1}$ is a martingale with respect to the filtration $(\Sigma_i)_{i \leq q+1}$.*

PROOF. Fix $j \leq q$. Let s be a bounded continuous function on $\mathbf{R}^{q+1}$ that depends only on $x_1, \dots, x_j$. For $a > 0$ and for a function f on $\mathbf{R}^{q+1}$ or on Ω , we denote by f^a its truncation at a and $-a$: that is, $f^a = (f \wedge a) \vee (-a)$. We have

$$\begin{aligned} \int x_{j+1} s(x) d\nu_q(x) &= \lim_{a \rightarrow \infty} \int x_{j+1}^a s(x) d\nu_q(x) \\ &= \lim_{a \rightarrow \infty} \lim_{i_1 \rightarrow \mathcal{U}} \lim_{i_2 \rightarrow \mathcal{U}} \dots \lim_{i_q \rightarrow \mathcal{U}} \int \phi_q^a(i_1, \dots, i_{j+1}) s(\phi_q, \dots, \phi_q(i_1, \dots, i_j)). \end{aligned}$$

We now note that we have $\|f - f^a\|_1 \rightarrow 0$ uniformly for $f \in E_A$, so we can intervert the limits and get

$$\begin{aligned} & \int x_{j+1} s(x) dv_q(x) \\ &= \lim_{i_1 \rightarrow \mathcal{U}} \cdots \lim_{i_q \rightarrow \mathcal{U}} \lim_{a \rightarrow \infty} \int \phi_q^a(i_1, \dots, i_{j+1}) s(\phi_q, \dots, \phi_q(i_1, \dots, i_j)) d\mu \\ &= \lim_{i_1 \rightarrow \mathcal{U}} \cdots \lim_{i_{j+1} \rightarrow \mathcal{U}} \int \phi_q(i_1, \dots, i_{j+1}) s(\phi_q, \dots, \phi_q(i_1, \dots, i_j)) d\mu. \end{aligned}$$

We now note that $s(\phi_q, \dots, \phi_q(i_1, \dots, i_j)) \in L^\infty \subset E^*$. Since

$$\text{weak-lim}_{i_{j+1} \rightarrow \mathcal{U}} \phi_q(i_1, \dots, i_j, i_{j+1}) = \phi_q(i_1, \dots, i_j),$$

we get

$$\begin{aligned} & \int x_{j+1} s(x) dv_q(x) \\ &= \lim_{i_1 \rightarrow \mathcal{U}} \cdots \lim_{i_j \rightarrow \mathcal{U}} \int \phi_q(i_1, \dots, i_j) s(\phi_q, \phi_q(i_1), \dots, \phi_q(i_1, \dots, i_j)) d\mu \\ &= \lim_{i_1 \rightarrow \mathcal{U}} \cdots \lim_{i_j \rightarrow \mathcal{U}} \lim_{a \rightarrow \infty} \int \phi_q^a(i_1, \dots, i_j) s(\phi_q, \phi_q(i_1), \dots, \phi_q(i_1, \dots, i_j)) d\mu \\ &= \lim_{a \rightarrow \infty} \lim_{i_1 \rightarrow \mathcal{U}} \cdots \lim_{i_j \rightarrow \mathcal{U}} \int \phi_q^a(i_1, \dots, i_j) s(\phi_q, \dots, \phi_q(i_1, \dots, i_j)) d\mu \\ &= \lim_{a \rightarrow \infty} \int x_j^a s(x_1, \dots, x_j) dv_q(x) \\ &= \int x_j s(x_1, \dots, x_j) dv_q(x). \end{aligned}$$

The lemma is proved.

The following elementary result is likely to be known; but we do not know a reference for it.

LEMMA 3. *Let $A > 0$, $\eta > 0$. Then there exists $N(A, \eta)$ depending only on A and η such that for any martingale $(X_j)_{j \leq q}$, with $\|X_j\|_1 \leq A$ for $j \leq q$, we have*

$$\text{card}\{j < q; P(\{|X_{j+1} - X_j| > \eta\}) > \eta\} \leq N(A, \eta).$$

PROOF. Since X_j is the difference of two positive martingales (Y_j) , (Z_j) for which $\|Y_j\|_1, \|Z_j\|_1 \leq A$, we can suppose that $X_j \geq 0$. Let $h = 3A/\eta$, so for each $j \leq q$ we have $P(\{X_j \geq h\}) \leq \eta/3$. Let $Y_j = h - X_j \wedge h$. Then $0 \leq Y_j \leq h$, and $E^j(Y_{j+1}) \geq Y_j$ for $1 \leq j < q$. We have

$$E((Y_{j+1} - Y_j)^2) \leq E(Y_{j+1}^2) + E(Y_j^2) - 2E(Y_j Y_{j+1}) \leq E(Y_{j+1}^2) - E(Y_j^2),$$

so $\sum_{j=1}^{q-1} E(Y_{j+1} - Y_j)^2 \leq h^2$. Let

$$A = \{j < q; E(Y_{j+1} - Y_j)^2 \geq \eta^2/3\},$$

so $\text{card } A \leq N(A, \eta) = 2h^2/\eta^2$. For $j \in A$ we have $P(\{|Y_{j+1} - Y_j| \geq \eta\}) \leq \eta/3$, so $P(\{|X_{i+1} - X_i| \geq \eta\}) \leq \eta$. The proof is complete.

We now prove the Main Fact.

Let a be as in Lemma 1. Since the unit ball of E is equi-integrable, there exists $\eta > 0$ such that for $x \in E$, $\|x\| \leq 2a$, we have

$$(18) \quad \mu(\{|x| > \eta\}) < \eta \Rightarrow \|x\|_1 \leq \varepsilon/a.$$

Let $N(A, \eta)$, as in Lemma 3, and let $r = N(A, \eta) + 1$. Let $q \geq r$, I be as in Lemma 1. We apply Lemma 3 to the martingale $(x_j)_{j \leq q+1}$ on $(\mathbf{R}^{q+1}, \nu_q)$, so we get $j \leq q$ such that $\nu_q(\{|x_{j+1} - x_j| < \eta\}) > 1 - \eta$.

Consider the set T of families $i_1 < \dots < i_j$ such that the following set J belongs to $\mathcal{U}$:

$$(19) \quad J = \left\{ i_{j+1} > i_j; \mu\left(\left|\phi_q(i_1, \dots, i_j) - \phi_q(i_1, \dots, i_{j+1})\right| < \eta\right) > 1 - \eta \right\}.$$

It follows from (3), (17) that ϕ_q is a weak cluster point of $\phi_q(i_1, \dots, i_j)$ for $(i_1, \dots, i_j)$ in T . So (15) shows that we can find a finite set S , for s in S numbers $\alpha_s \geq 0$, with $\sum \alpha_s = 1$ and elements $(i_1^s, \dots, i_j^s)$ of T such that

$$\begin{aligned} \left\| \sum \alpha_s \phi_q(i_1^s, \dots, i_j^s) - \phi_q \right\|_E &\leq \varepsilon, \\ \int \xi \psi \phi_q(i_1^s, \dots, i_j^s) d\mu &> b', \end{aligned}$$

where $\xi = \text{sign } \psi \phi$. It follows from (15) that for each $n \in I$, there is $s \in S$ such that

$$\int_{\{|\psi_n| > a\}} |\psi_n \phi_q(i_1^s, \dots, i_j^s)| d\mu \leq 2\varepsilon.$$

So, there is $i_1 < \dots < i_j$ in T such that

$$\int \xi \psi \phi_q(i_1, \dots, i_j) d\mu > b'$$

and $I' \in \mathcal{U}$, where

$$(20) \quad I' = \left\{ n \in I; \int_{\{|\psi_n| > a\}} |\psi_n \phi_q(i_1, \dots, i_j)| d\mu \leq 2\varepsilon \right\}.$$

Since $\psi = \text{weak-lim}_{n \rightarrow \mathcal{U}} \psi_n$, we can assume that for n in I' we have $\int \xi \psi_n \phi_q(i_1, \dots, i_j) d\mu > b'$. Together with (20), this implies

$$(21) \quad \int_{\{|\psi_n| \leq a\}} \xi \psi_n \phi_q(i_1, \dots, i_j) d\mu > b' - 2\varepsilon.$$

Since $J \in \mathcal{U}$, $I' \in \mathcal{U}$, we can pick $n \in I'$ such that (21) holds and

$$(22) \quad \mu\left(\left|\phi_q(i_1, \dots, i_j, n) - \phi_q(i_1, \dots, i_{j+1})\right| < \eta\right) > 1 - \eta.$$

From (18) and (22) follows that

$$\left\| \phi_q(i_1, \dots, i_j, n) - \phi_q(i_1, \dots, i_j) \right\|_1 \leq \varepsilon/a,$$

so we get from (21) that

$$\int_{\{|\psi_n| \leq a\}} \xi \psi_n \phi_q(i_1, \dots, i_j, n) d\mu > b' - 3\varepsilon.$$

On the other hand, we have

$$\begin{aligned}
 & \left| \int_{\{|\psi_n| > a\}} \psi_n \phi_q(i_1, \dots, i_j, n) d\mu \right| \\
 & \geq \left| \int \psi_n (\phi_q(i_1, \dots, i_j, n) - \phi_q(i_1, \dots, i_j)) d\mu \right| \\
 & \quad - \left| \int_{\{|\psi_n| \leq a\}} \psi_n (\phi_q(i_1, \dots, i_j, n) - \phi_q(i_1, \dots, i_j)) d\mu \right| \\
 & \quad - \left| \int_{\{|\psi_n| > a\}} \psi_n \phi_q(i_1, \dots, i_j) d\mu \right|.
 \end{aligned}$$

The first term is ≥ 1 from (5), the second is $\leq \varepsilon$ from (18) and (22), and the third is $\leq 2\varepsilon$, since $n \in I'$. Finally, we have shown that

$$\int |\psi_n \phi_q(i_1, \dots, i_j, n)| d\mu > 1 + b' - 6\varepsilon > b + 1.$$

The proof is complete.

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