DECOMPOSITION OF CHARGES AND MEASURES

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ABSTRACT. By means of a characterization of the band generated by a set of charges on an algebra of sets one can obtain unified proofs of some decomposition theorems for charges and measures.

1. Introduction. Let $\mathfrak{A}$ be an algebra of subsets of a set Ω . By $\text{ba}(\mathfrak{A})$ we denote the order complete vector lattice of all bounded, additive, real functions on $\mathfrak{A}$ and elements of its positive cone $\text{ba}(\mathfrak{A})_+$ are referred to as charges. Let V be an order complete vector sublattice of $\text{ba}(\mathfrak{A})$. We prove a decomposition theorem for charges in V which implies the following decomposition. Let M be the set of all charges $\lambda \in V_+$ such that λ is an extreme point of the convex set of charges in V which coincide with λ on a subalgebra of $\mathfrak{A}$. Then each charge $\mu \in V_+$ has a unique decomposition $\mu = \mu_1 + \mu_2$, where μ_1 is a countable sum of elements of M and μ_2 is a charge which is disjoint from each element of M . However, it is shown that this decomposition only depends on a certain property of the set M and a suitable characterization of the band generated by M . For several choices of V the elements of M may be characterized by an approximation property [3, 4]. In case $V = \text{ba}(\mathfrak{A})$ and $V = \text{ca}(\mathfrak{A})$ we derive characterizations of the second component in the above decomposition, where $\text{ca}(\mathfrak{A})$ denotes the order complete vector sublattice of $\text{ba}(\mathfrak{A})$ consisting of the countably additive functions in $\text{ba}(\mathfrak{A})$; elements of $\text{ca}(\mathfrak{A})_+$ are referred to as measures.

We need some further notation. For a subset M of V let $B(M)$ denote the band generated by M in V and $M^\perp$ the set of all $\mu \in V$ disjoint from each $\lambda \in M$, i.e. $|\mu| \wedge |\lambda| = 0$ for each $\lambda \in M$. For a convex subset C of V let $\text{ex } C$ denote the set of all extreme points of C . If $\nu, \mu \in V_+$ with $\nu \leq \mu$, then ν is called a minorant of μ .

2. The decomposition theorem. Let M be a subset of V_+ with $0 \in M$. Define

$$N_1 = \left\{ \mu \in V_+ : \mu = \sum_{n=1}^{\infty} \lambda_n, \lambda_n \in M \text{ for every } n \in \mathbb{N} \right\}$$

and

$$N_2 = \{ \mu \in V_+ : \mu \text{ admits no nonzero minorant in } M \}.$$

LEMMA 1. *Each $\mu \in V_+$ has a decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in N_1$ and $\mu_2 \in N_2$.*

Received by the editors May 25, 1984 and, in revised form, February 1, 1985.

1980 *Mathematics Subject Classification.* Primary 28A10.

Key words and phrases. Bands generated by a set of charges, extreme extensions of charges and measures.

PROOF. Let $\mu \in V_+$ and consider the nonempty set

$$H = \left\{ f \in M^{\mathbf{R}} : \sum_{x \in \mathbf{R}} f(x) \leq \mu \right\}.$$

For $f \in H$ the support set $S(f) = \{f \neq 0\}$ is countable, which implies $\sum_{x \in \mathbf{R}} f(x) \in N_1$. Define an ordering on H by $f \leq g$ if and only if $f(x) = g(x)$ for every $x \in S(f)$. Then H is inductively ordered. In fact, for a chain H_0 in H the map f_0 defined by $f_0(x) = f(x)$ if $x \in S(f)$ for some $f \in H_0$, and $f_0(x) = 0$ otherwise, is an upper bound of H_0 in H . Thus by the Kuratowski-Zorn lemma there exists a maximal element f_1 of H . Defining $\mu_1 = \sum_{x \in \mathbf{R}} f_1(x)$ and $\mu_2 = \mu - \mu_1$ we obtain the required decomposition.

Consider the following property of M :

$$(P) \quad \mu \in V_+, \lambda \in M, \mu \leq \lambda \text{ imply } \mu \in N_1.$$

LEMMA 2. *The following statements are equivalent:*

- (i) (P) is satisfied.
- (ii) $(M^\perp)_+ = N_2$.
- (iii) $B(M)_+ = N_1$.

PROOF. (i) $\Rightarrow$ (ii). The inclusion $(M^\perp)_+ \subset N_2$ is obvious. Conversely, let $\mu \in N_2$ and $\lambda \in M$. By (i) $\lambda \wedge \mu \in N_1$ holds, i.e. $\lambda \wedge \mu = \sum_{n=1}^\infty \lambda_n$ with $\lambda_n \in M$ for every $n \in \mathbf{N}$. Since $\lambda_n \leq \lambda \wedge \mu \leq \mu$, we obtain $\lambda_n = 0$ for every $n \in \mathbf{N}$ and so $\lambda \wedge \mu = 0$. This yields $\mu \in M^\perp$.

(ii) $\Rightarrow$ (iii). Clearly $B(M)_+ \supset N_1$ holds. Conversely, let $\mu \in B(M)_+$. By Lemma 1 and (ii) μ has a decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in N_1$ and $\mu_2 \in (M^\perp)_+$. Since $\mu_2 \leq \mu$, we obtain $\mu_2 \in B(M)$, hence $\mu_2 = 0$. This yields $\mu \in N_1$.

(iii) $\Rightarrow$ (i). Obvious.

Now one obtains from the Riesz decomposition theorem (cf. [1, 1.5.10]) the following decomposition.

THEOREM. *Assume that (P) is satisfied. Then each $\mu \in V_+$ has a unique decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in N_1$ and $\mu_2 \in N_2$.*

REMARK. In case $V = \text{ca}(\mathfrak{A})$, where $\mathfrak{A}$ is a σ -algebra on Ω , consider the following property of M which is stronger than (P):

$$(P') \quad \mu \in \text{ca}(\mathfrak{A})_+, \lambda \in M, \mu \ll \lambda \text{ imply } \mu \in M.$$

Then, under (P'), for $\mu \in \text{ca}(\mathfrak{A})_+$ there exists a countable partition $\{A_n: n \in I \cup \{0\}\}$, $I \subset \mathbf{N}$ of Ω in $\mathfrak{A}$, such that for the components of μ in the above decomposition $\mu_1 = \sum_{n \in I} \mu(A_n \cap \cdot)$ with $\mu(A_n \cap \cdot) \in M$ for every $n \in I$ and $\mu_2 = \mu(A_0 \cap \cdot)$ holds. To see this let

$$\mathfrak{D} = \{A \in \mathfrak{A}: \mu(A) > 0, \mu(A \cap \cdot) \in M\}.$$

The subsets of $\mathfrak{D}$ consisting of pairwise disjoint sets are inductively ordered by inclusion, so by the Kuratowski-Zorn lemma there is a maximal subset $\{A_n: n \in I\}$ of this type. Since it is clearly countable, let $I \subset \mathbf{N}$ and $A_0 = (\bigcup_{n \in I} A_n)^c$. If $\lambda \in M$, $\lambda \leq \mu(A_0 \cap \cdot)$, and f denotes a $\mu(A_0 \cap \cdot)$ -density of λ , then

$$\mu(A_0 \cap \{f > 0\} \cap \cdot) \ll \lambda.$$

By (P') and the maximality of $\{A_n: n \in I\}$ this implies $\lambda = 0$. Thus we obtain $\mu(A_0 \cap \cdot) \in N_2$.

3. Examples. (1) Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$. For $\mu \in V_+$ we set $E(\mu|\mathfrak{B}) = \{\nu \in V_+: \nu|\mathfrak{B} = \mu|\mathfrak{B}\}$. Let

$$M = \{\lambda \in V_+: \lambda \in \text{ex } E(\lambda|\mathfrak{B})\}.$$

Then $0 \in M$ and (P) is satisfied. In fact, if $\mu \in V_+$ is a minorant of $\lambda \in M$ and $\mu = (\mu_1 + \mu_2)/2$, where $\mu_i \in E(\mu|\mathfrak{B})$, $i = 1, 2$, then

$$\lambda = (\mu_1 + \lambda - \mu + \mu_2 + \lambda - \mu)/2$$

and $\mu_i + \lambda - \mu \in E(\lambda|\mathfrak{B})$, $i = 1, 2$. It follows $\lambda = \mu_1 + \lambda - \mu = \mu_2 + \lambda - \mu$, thus $\mu_1 = \mu_2$. This implies $\mu \in M$. Therefore the Theorem applies.

In order to characterize M and the corresponding set N_2 we need the following information. For $\mu \in \text{ba}(\mathfrak{A})_+$ define the $\mathbb{R}^{\mathfrak{B}}$ -valued vector charge $\hat{\mu}$ on $\mathfrak{A}$ by $\hat{\mu} = (\mu(B \cap \cdot))_{B \in \mathfrak{B}}$. $\hat{\mu}$ is said to be strongly continuous if for each $\varepsilon > 0$ there exists a finite partition $\{A_1, \dots, A_n\}$ of Ω in $\mathfrak{A}$ such that $\hat{\mu}(A_i) \leq \varepsilon \hat{\mu}(\Omega)$ for every i ; $\hat{\mu}$ is said to be strongly nonatomic if for each $A \in \mathfrak{A}$ and $\alpha \in [0, 1]$ there exists $A_1 \in \mathfrak{A}$ such that $A_1 \subset A$ and $\hat{\mu}(A_1) = \alpha \hat{\mu}(A)$. A charge μ on $\mathfrak{A}$ is called $\mathfrak{B}$ -approximable if $\mathfrak{B}$ is dense in $\mathfrak{A}$ in the topology of the semimetric $(A, B) \mapsto \mu(A \Delta B)$.

LEMMA 3. *Let μ be a charge on $\mathfrak{A}$. Consider the following conditions:*

- (i) μ has no nonzero $\mathfrak{B}$ -approximable minorant in $\text{ba}(\mathfrak{A})_+$.
- (ii) $\mu(A \cap \cdot)$ is not $\mathfrak{B}$ -approximable for every $A \in \mathfrak{A}$ with $\mu(A) > 0$.
- (iii) $\hat{\mu}$ is a Liapunov vector charge, i.e. $\{\hat{\mu}(C): C \in \mathfrak{A}, C \subset A\}$ is a convex subset of $\mathbb{R}^{\mathfrak{B}}$ for every $A \in \mathfrak{A}$.
- (iv) $\hat{\mu}$ is strongly nonatomic.
- (v) $\hat{\mu}$ is strongly continuous.

Then (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are valid. If, in addition, $\mathfrak{A}$ is a σ -algebra and μ a measure, then these conditions are all equivalent.

PROOF. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious and (iv) $\Rightarrow$ (v) may be proved similar to the scalar case.

(v) $\Rightarrow$ (i). Let $\nu \in \text{ba}(\mathfrak{A})_+$ be $\mathfrak{B}$ -approximable, $\nu \leq \mu$ and let $\varepsilon > 0$. By assumption there is a partition $\{A_1, \dots, A_n\}$ of Ω in $\mathfrak{A}$ such that $\mu(A_i \cap B) \leq \varepsilon \mu(B)$ for every i and $B \in \mathfrak{B}$ and for each i there exists $B_i \in \mathfrak{B}$ such that $\nu(A_i \Delta B_i) \leq \varepsilon/n^2$. Then

$$\nu\left(\left(\bigcup_{i=1}^n B_i\right)^c\right) \leq \sum_{i=1}^n \nu(A_i \Delta B_i) \leq \varepsilon/n$$

and hence, $\nu(\Omega) \leq \nu(\bigcup_{i=1}^n B_i) + \varepsilon/n$. Furthermore, for each pair of distinct indices i, j we have

$$\nu(B_i \cap B_j) \leq \nu(A_i^c \cap B_i) + \nu(A_j^c \cap B_j) \leq 2\varepsilon/n^2.$$

For the partition $\{C_1, \dots, C_n\}$ of $\bigcup_{i=1}^n B_i$ in $\mathfrak{B}$ defined by $C_1 = B_1$ and $C_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i$ for $k = 2, \dots, n$ this yields

$$\nu(B_k \setminus C_k) \leq \sum_{i=1}^{k-1} \nu(B_k \cap B_i) \leq 2\varepsilon/n$$

for every $k \geq 2$. Thus we obtain $\nu(B_i) \leq \varepsilon\mu(C_i) + 3\varepsilon/n$ for every i and so $\nu(\Omega) \leq \varepsilon\mu(\Omega) + 4\varepsilon$. As $\varepsilon > 0$ is arbitrary, $\nu = 0$ holds.

(ii) $\Leftrightarrow$ (iii). Now suppose that $\mathfrak{A}$ is a σ -algebra and μ a measure on $\mathfrak{A}$. According to [2, Corollary], (iii) holds if and only if $\{1_B: B \in \mathfrak{B}\}$ is a thin subset of $L_1(\mu)$ in the sense of [2] and by the Hahn-Banach theorem this is equivalent to the fact that $S(\mathfrak{B})$ is not a norm-dense subset of $L_1(\mu(A \cap \cdot))$ for every $A \in \mathfrak{A}$ with $\mu(A) > 0$, where $S(\mathfrak{B})$ is the vector space of all $\mathfrak{B}$ -simple real functions. To prove that the latter property is equivalent to (ii) we have to show that a measure ν on $\mathfrak{A}$ is $\mathfrak{B}$ -approximable if and only if $S(\mathfrak{B})$ is dense in $L_1(\nu)$. The “only if” part is obvious. To prove the “if” part let $A \in \mathfrak{A}$ and $\varepsilon > 0$. There is a sequence in $S(\mathfrak{B})$ which converges to 1_A in $L_1(\nu)$. Since $L_1(\nu|_{\mathfrak{B}_0})$ is a Banach space, where $\mathfrak{B}_0$ denotes the σ -algebra generated by $\mathfrak{B}$, there is a function $f \in L_1(\nu|_{\mathfrak{B}_0})$ such that $\int |1_A - f| d\nu = 0$. For $B_0 = \{f = 1\}$ we obtain $B_0 \in \mathfrak{B}_0$ and $\nu(A \Delta B_0) = 0$. If we choose $B_1 \in \mathfrak{B}$ with $\nu(B_0 \Delta B_1) \leq \varepsilon$, then $\nu(A \Delta B_1) \leq \varepsilon$ holds.

Part (a) of the following lemma is a generalization of [1, Proposition 5.3.7].

LEMMA 4. *Assume that $\mathfrak{B}$ is a finite subalgebra of $\mathfrak{A}$.*

(a) *Let $\mathfrak{A}$ be a σ -algebra, $\mathcal{C}$ an algebra generating $\mathfrak{A}$, and μ a measure on $\mathfrak{A}$. Then $\hat{\mu}$ is strongly continuous if and only if $\hat{\mu}|_{\mathcal{C}}$ is strongly continuous.*

(b) *Let μ be a charge on $\mathfrak{A}$. Then μ has no nonzero $\mathfrak{B}$ -approximable minorant in $\text{ba}(\mathfrak{A})_+$ if and only if $\hat{\mu}$ is strongly continuous.*

PROOF. (a) We have to prove the “only if” part. We may assume $\mu(\Omega) > 0$. Let $\varepsilon > 0$. Since $\hat{\mu}$ is strongly continuous, there is a partition $\{A_1, \dots, A_n\}$ of Ω in $\mathfrak{A}$ such that $\hat{\mu}(A_i) \leq \varepsilon\hat{\mu}(\Omega)/2$ for every i . Let $\delta = \min\{\varepsilon\mu(B)/2: B \in \mathfrak{B}, \mu(B) > 0\}$. Then $\delta > 0$. For each i there exists $C_i \in \mathcal{C}$ such that $\mu(A_i \Delta C_i) \leq \delta/n$. Then

$$\mu\left(\left(\bigcup_{i=1}^n C_i\right)^c\right) \leq \sum_{i=1}^n \mu(A_i \Delta C_i) \leq \delta$$

and, hence, $\hat{\mu}((\bigcup_{i=1}^n C_i)^c) \leq \varepsilon\hat{\mu}(\Omega)$. Defining a partition $\{D_1, \dots, D_{n+1}\}$ of Ω in $\mathcal{C}$ by $D_1 = C_1$, $D_k = C_k \setminus \bigcup_{i=1}^{k-1} C_i$ for $k = 2, \dots, n$ and $D_{n+1} = (\bigcup_{i=1}^n C_i)^c$, we see that $\hat{\mu}(D_k) \leq \varepsilon\hat{\mu}(\Omega)$ holds for every k . This completes the proof of (a).

(b) The “if” part is contained in Lemma 3. The “only if” part follows from (a) and Lemma 3 by transferring the assertion to the Stone space of $\mathfrak{A}$.

(2) Choose in (1) $V = \text{ca}(\mathfrak{A})$, where $\mathfrak{A}$ is a σ -algebra on Ω . Then $M = \{\lambda \in \text{ca}(\mathfrak{A})_+: \lambda \text{ is } \mathfrak{B}\text{-approximable}\}$ (cf. [4]) and, by Lemma 3, $N_2 = \{\mu \in \text{ca}(\mathfrak{A})_+: \hat{\mu} \text{ is a Liapunov vector measure}\}$. Note that (P') is satisfied for M and so the Remark applies. If, in particular, $\mathfrak{B} = \{\emptyset, \Omega\}$, then $M = \{\lambda \in \text{ca}(\mathfrak{A})_+: \lambda \text{ is at most two-valued}\}$ and $N_2 = \{\mu \in \text{ca}(\mathfrak{A})_+: \mu \text{ is nonatomic}\}$.

(3) Choose in (1) $V = \text{ba}(\mathfrak{A})$. Then according to [4, Theorem 1] $M = \{\lambda \in \text{ba}(\mathfrak{A})_+: \lambda \text{ is } \mathfrak{B}\text{-approximable}\}$. If $\mathfrak{B}$ is finite, we obtain from Lemma 4(b) $N_2 = \{\mu \in \text{ba}(\mathfrak{A})_+: \hat{\mu} \text{ is strongly continuous}\}$. In particular, if $\mathfrak{B} = \{\emptyset, \Omega\}$, then $M = \{\lambda \in \text{ba}(\mathfrak{A})_+: \lambda \text{ is at most two-valued}\}$ and $N_2 = \{\mu \in \text{ba}(\mathfrak{A})_+: \mu \text{ is strongly continuous}\}$, where a charge μ is said to be strongly continuous if for each $\varepsilon > 0$ there exists a partition $\{A_1, \dots, A_n\}$ of Ω in $\mathfrak{A}$ such that $\mu(A_i) \leq \varepsilon$ for every i . Thus from the Theorem follows the Sobczyk-Hammer decomposition (cf. [1, Theorem 5.2.7]).

(4) Let G be a semigroup which acts from the left on Ω and assume that $\mathfrak{A}$ and $\mathfrak{B}$ are invariant algebras, $\mathfrak{B} \subset \mathfrak{A}$. Let $\text{ba}(\mathfrak{A})^G$ denote the order complete vector sublattice of $\text{ba}(\mathfrak{A})$ consisting of the invariant elements of $\text{ba}(\mathfrak{A})$. Choose in (1) $V = \text{ba}(\mathfrak{A})^G$ or $V = \text{ca}(\mathfrak{A})^G$. An invariant charge μ on $\mathfrak{A}$ is called $(\mathfrak{B}, G)$ -approximable if for each sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{A}$ such that $\lim_{n, m \rightarrow \infty} \mu(A_n \Delta A_m) = 0$ and $\lim_{n \rightarrow \infty} \mu(A_n \Delta g^{-1}A_n) = 0$ for every $g \in G$ and for each $\varepsilon > 0$ there exists $B \in \mathfrak{B}$ with $\inf_{n \in \mathbb{N}} \mu(A_n \Delta B) \leq \varepsilon$. Then according to [3] we obtain $M = \{\lambda \in V_+ : \lambda \text{ is } (\mathfrak{B}, G)\text{-approximable}\}$. If $\mathfrak{A}$ is an invariant σ -algebra, then an invariant measure μ on $\mathfrak{A}$ is $(\mathfrak{B}, G)$ -approximable if and only if for each $A \in \mathfrak{A}(\mu)^G$ and each $\varepsilon > 0$ there exists $B \in \mathfrak{B}$ such that $\mu(A \Delta B) \leq \varepsilon$, where $\mathfrak{A}(\mu)^G = \{A \in \mathfrak{A} : \mu(A \Delta g^{-1}A) = 0 \text{ for every } g \in G\}$. In particular, if $\mathfrak{B} = \{\emptyset, \Omega\}$ and $V = \text{ca}(\mathfrak{A})^G$, then $M = \{\lambda \in \text{ca}(\mathfrak{A})_+^G : \lambda \text{ is ergodic}\}$.

(5) Let $V = \text{ca}(\mathfrak{A})$, where $\mathfrak{A}$ is a σ -algebra and let $\mathcal{C}$ be a σ -algebra with $\mathfrak{A} \subset \mathcal{C}$. For $\mu \in \text{ca}(\mathfrak{A})_+$ set $E(\mu) = \{\nu \in \text{ca}(\mathcal{C})_+ : \nu|_{\mathfrak{A}} = \mu\}$. Let

$$M = \{\lambda \in \text{ca}(\mathfrak{A})_+ : \text{ex } E(\lambda) \neq \emptyset\}.$$

Then $M = \{\lambda \in \text{ca}(\mathfrak{A})_+ : \text{there exists } \nu \in E(\lambda) \text{ which is } \mathfrak{A}\text{-approximable}\}$ (cf. (2)), $0 \in M$, and (P') is satisfied. In fact, let $\mu \in \text{ca}(\mathfrak{A})_+$ and $\lambda \in M$ such that $\mu \ll \lambda$. For $\nu \in \text{ex } E(\lambda)$ define $\rho \in \text{ca}(\mathcal{C})_+$ by $\rho(C) = \int_C (d\mu/d\lambda) d\nu$, $C \in \mathcal{C}$. Then $\rho \in E(\mu)$, $\rho \ll \nu$ and, hence, $\rho \in \text{ex } E(\mu)$. This implies $\mu \in M$. In contrast to (2) now $M = N_1$ holds. To see this let $\mu \in N_1$. By the Remark there exists a partition $\{A_n : n \in I\}$, $I \subset \mathbb{N}$ of Ω in $\mathfrak{A}$, such that $\mu = \sum_{n \in I} \mu(A_n \cap \cdot)$ and $\mu(A_n \cap \cdot) \in M$ for every $n \in I$. Let $\nu_n \in \text{ex } E(\mu(A_n \cap \cdot))$ for $n \in I$, $\nu = \sum_{n \in I} \nu_n$, and $C \in \mathcal{C}$. Then $\nu_n = \nu_n(A_n \cap \cdot)$ for every $n \in I$, $\nu \in E(\mu)$, and for each $n \in I$ there exists $B_n \in \mathfrak{A}$ such that $\nu_n(C \Delta B_n) = 0$. Defining $A \in \mathfrak{A}$ by $A = \bigcup_{n \in I} (A_n \cap B_n)$ we obtain $\nu(C \Delta A) = 0$. This implies $\nu \in \text{ex } E(\mu)$ and, hence, $\mu \in M$.

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