DEFINABLE AUTOMORPHISMS OF $\mathcal{P}(\omega)/\text{fin}$

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ABSTRACT. We investigate definable automorphisms of $\mathcal{P}(\omega)/\text{fin}$ and show that e.g. every Borel automorphism is trivial. The existence of nontrivial projective automorphisms is consistent and independent from ZFC + CH.

Throughout this paper $\mathcal{P}(\omega)/\text{fin}$ will denote the Boolean algebra of $\mathcal{P}(\omega)$ modulo the ideal of finite sets. Call an automorphism F of $\mathcal{P}(\omega)/\text{fin}$ trivial if there are two cofinite sets $a, b \subseteq \omega$ and a bijection $f: a \rightarrow b$ such that $F[x] = [f''(a \cap x)]$. Here $[x]$ denotes the equivalence class of x under the equivalence relation $x \equiv_* y$ iff $(x \setminus y) \cup (y \setminus x)$ is finite. Note that there are exactly $2^{\aleph_0}$ trivial automorphisms. It is a well-known fact, first proved by W. Rudin [7], that the Continuum Hypothesis implies there are $2^{2^{\aleph_0}}$ automorphisms of $\mathcal{P}(\omega)/\text{fin}$, hence most of them are nontrivial. It is also consistent with $\neg\text{CH}$ that there is a nontrivial automorphism of $\mathcal{P}(\omega)/\text{fin}$. This was proved by Baumgartner (unpublished). He noticed that if there is a family $\{a_\alpha: \alpha < \omega_1\}$ of subsets of ω such that

- (i) $a_\alpha \setminus a_\beta$ is finite for every $\alpha < \beta < \omega_1$,
- (ii) $\{a_\alpha: \alpha < \omega_1\} \cup [\omega]^{<\omega}$ generates a maximal ideal I ,

then one can build by induction a family $\{f_\alpha: \alpha < \omega_1\}$ of functions such that

- (iii) $f_\alpha: a_\alpha \rightarrow a_\alpha$ is a permutation without fixed points,
- (iv) if $\alpha < \beta$ then $f_\beta \upharpoonright a_\alpha = * f_\alpha$.

Then we simply define $F: \mathcal{P}(\omega)/\text{fin} \rightarrow \mathcal{P}(\omega)/\text{fin}$ by

$$F[x] = \begin{cases} [f_\alpha''(x)] & \text{if } x \in I \text{ and } x \subseteq a_\alpha \text{ for some } \alpha < \omega_1, \\ [\omega \setminus f_\alpha''(\omega \setminus x)] & \text{if } x \notin I \text{ and } \omega \setminus x \subseteq a_\alpha \text{ for some } \alpha < \omega_1. \end{cases}$$

Now, (iv) ensures that F is well defined, (ii) that it is defined everywhere and (iii) that it is nontrivial. To see that, assume F is induced by some $f: a \xrightarrow[onto]{1-1} b$ for $a, b \subseteq \omega$ cofinite. One can easily find disjoint infinite sets a^0, a^1 and a^2 such that $a^0 \cup a^1 \cup a^2 = a$ and $f(a^\varepsilon) \cap a^\varepsilon = \emptyset$ for $\varepsilon = 0, 1, 2$. Let $\varepsilon < 3$ be such that $a^\varepsilon \notin I$. It follows from the definition of F that $f''(a^\varepsilon) \notin I$, but, since I is a maximal ideal, $f''(a^\varepsilon)$ and a^ε must intersect which is a contradiction.

The existence of a family satisfying (i) and (ii) is consistent with $\neg\text{CH}$ (see Kunen [3, p. 289]).

Using the method of Laver [4] we can prove the consistency of “ZFC + MA(σ -centered) + $\neg\text{CH}$ + there is a nontrivial automorphism of $\mathcal{P}(\omega)/\text{fin}$ ”. It is

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not known to the author if $\text{MA} + \neg\text{CH}$ implies, or is at least consistent with, the existence of nontrivial automorphisms.

On the other hand, S. Shelah [9] proved the following.

THEOREM 1. *If ZF is consistent then so is “ZFC + $2^{\aleph_0} = \aleph_2$ + every automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial”.*

Here, we shall be interested in definable nontrivial automorphisms of $\mathcal{P}(\omega)/\text{fin}$. Throughout the rest of the paper we identify $\mathcal{P}(\omega)$ and 2^ω , i.e. $x \subseteq \omega$ is identified with its characteristic function χ_x . For a pointclass Γ we say that F is a Γ -automorphism if $\text{Gr}(F) \in \Gamma$, where $\text{Gr}(F) = \{\langle x, y \rangle : F[x] = [y]\}$. The following lemma is a distilled form of Shelah's Main Lemma from [9, p. 131].

LEMMA 2. *Let $F \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$. Suppose there is a dense G_δ subset X of 2^ω and a continuous $\varphi: X \rightarrow 2^\omega$ such that $\forall x \in X F[x] = [\varphi(x)]$. Then F is trivial.*

PROOF. Let us first assume $X = 2^\omega$. For $s \in 2^{<\omega}$ let $N_s = \{x \in 2^\omega : s \subseteq x\}$ be the basic clopen set determined by s . For $s, t \in 2^{<\omega}$ say that s forces t if $\varphi(N_s) \subseteq N_t$. We build by induction an increasing sequence of integers $\langle n_i : i < \omega \rangle$ and functions $f_i: [n_i, n_{i+1}) \rightarrow 2$ such that

- (i) for every $i < \omega$ and every $s \in 2^{n_i}$, $s \cup f_i$ forces some $t \in 2^{n_i}$,
- (ii) for every $i < \omega$, $s, s' \in 2^{n_i}$, every $k > n_{i+1}$ and $g: [n_{i+1}, k) \rightarrow 2$ if $s \cup f_i \cup g$ forces some t and $s' \cup f_i \cup g$ forces some t' then $\forall i \geq n_{i+1} t(i) = t'(i)$ if both are defined.

Assume f_{i-1} and n_i are defined. We have to construct f_i and n_{i+1} . First, we take care of (i).

Enumerate 2^{n_i} as $\{s_j : j < 2^{n_i}\}$ and construct partial functions f_i^j for $j \leq 2^{n_i}$ by induction. Set $f_i^0 = f_{i-1}$. Suppose we have f_i^j and it satisfies $\text{dom}(f_i^j) \cap n_i = \emptyset$. Since φ is continuous we can find $f_i^{j+1} \supseteq f_i^j$ such that $s_j \cup f_i^{j+1}$ is a function which forces some $t \in 2^{n_i}$. Finally let $g_i = f_i^{2^{n_i}}$. It is clear that if $f \supseteq g_i$ is such that $\text{dom}(f) \cap n_i = \emptyset$ then f satisfies (i) in place of f_i .

To take care of (ii) we need an auxiliary definition. For $s, s' \in 2^{n_i}$ call a partial function f a *witness* for $\langle s, s' \rangle$ if there is some $n > n_i$ such that

- (iii) $f: [n_i, n) \rightarrow 2$,
- (iv) whenever $k > n$ and $g: [n, k) \rightarrow 2$, t, t' are in $2^{<\omega}$ such that $s \cup f \cup g$ forces t and $s' \cup f \cup g$ forces t' , then t and t' agree above n , i.e. if

$$\sigma = (\text{dom } t \cap \text{dom } t') \setminus n$$

then $t \upharpoonright \sigma = t' \upharpoonright \sigma$.

Claim. *For every partial function $\tau: [n_i, k) \rightarrow 2$ and every $s, s' \in 2^{n_i}$ there is a partial function $f \supseteq \tau$ which is a witness for $\langle s, s' \rangle$.*

PROOF. Assume otherwise. We can produce sequences of partial functions $\langle \tau^j : j < \omega \rangle$, $\langle t^j : j < \omega \rangle$ and $\langle t'^j : j < \omega \rangle$ and an increasing sequence of integers $\langle l^j : j < \omega \rangle$ such that $s \cup \tau^j$ forces t^j , $s' \cup \tau^j$ forces t'^j and $t^j(l^j) \neq t'^j(l^j)$. But then let $x = s \cup \bigcup \{\tau^j : j < \omega\}$ and $x' = s' \cup \bigcup \{\tau^j : j < \omega\}$.

We have $x = * x'$ while $\varphi(x) \neq * \varphi(x')$, a contradiction. The claim is proved.

To ensure (ii) we apply the claim repeatedly for each pair $\langle s, s' \rangle \in 2^{n_i} \times 2^{n_i}$.

Enumerate $2^{n_i} \times 2^{n_i}$ as $\{\langle s^j, s'^j \rangle: j < 2^{2^{n_i}}\}$ and construct $\langle g_i^j: j \leq 2^{2^{n_i}} \rangle$ and $\langle n_i^j: j \leq 2^{2^{n_i}} \rangle$ by induction. Set $g_i^0 = g_i$ and $n_i^0 = n_i$. Given g_i^j and n_i^j find $n_i^{j+1} > n_i^j$ and g_i^{j+1} using the claim such that $g_i^{j+1} \supseteq g_i^j$, $g_i^{j+1}: [n_i, n_i^{j+1}) \rightarrow 2$ and g_i^{j+1} is a witness for $\langle s^j, s'^j \rangle$. Finally, we let $f_{i+1} = g^{2^{2^{n_i}}}$ and $n_i = n_i^{2^{2^{n_i}}}$. This finishes the inductive construction of the f_i 's and n_i 's.

Now, let $a^\varepsilon = \bigcup\{[n_i, n_{i+1}): i \equiv \varepsilon \pmod{3}\}$ and $f^\varepsilon = \bigcup\{f_i: i \equiv \varepsilon \pmod{3}\}$ for $\varepsilon = 0, 1, 2$. Then consider the function $\varphi^0(x) = \varphi(x \cup a^1 \cup a^2) \setminus \varphi(a^1 \cup a^2)$. Since F is a homomorphism we have $\forall x \subseteq a^0$, $\varphi^0(x) = \ast \varphi(x)$. It is easy to see that there are functions h_i such that

- (v) $h_i: \mathcal{P}([n_{3i}, n_{3i+1})) \rightarrow \mathcal{P}([n_{3i-1}, n_{3i+2}))$ for $i < \omega$,
- (vi) $\varphi^0(x) = \bigcup\{h_i(x \cap [n_{3i}, n_{3i+1})): i < \omega\}$ for $x \subseteq a^0$.

Now, for almost all $i < \omega$ we must have

- (vii) $\forall u, v \in \mathcal{P}([n_{3i}, n_{3i+1})), h_i(u \cup v) = h_i(u) \cup h_i(v)$.

Assume there is an infinite $A \subseteq \omega$ such that for every $i \in A$ there are $u_i, v_i \subseteq [n_{3i}, n_{3i+1})$ and $h_i(u_i \cup v_i) \neq h_i(u_i) \cup h_i(v_i)$. Then let $u = \bigcup\{u_i: i \in A\}$ and $v = \bigcup\{v_i: i \in A\}$. Then $\varphi(u \cup v) \neq \ast \varphi(u) \cup \varphi(v)$ which contradicts the fact that F is a homomorphism. Similarly we get that h_i maps singletons to singletons for almost all $i < \omega$. For, assume otherwise. Let $A \subseteq \omega$ be infinite and $l_i \in [n_{3i}, n_{3i+1})$ for $i \in A$ such that $|h_i(\{l_i\})| \geq 2$ for every $i \in A$. Pick $m_i \in h_i(\{l_i\})$ for $i \in A$ and let $y = \{m_i: i \in A\}$. Then, clearly, there is no x such that $\varphi(x) = \ast y$. This contradicts the fact that F is onto.

So we can define a function $h^0: a^0 \rightarrow \omega$ by

$$h^0(l) = k \quad \text{if } l \in [n_{3i}, n_{3i+1}) \quad \text{and} \quad g_i(\{l\}) = \{k\}.$$

Then h^0 induces F on $\mathcal{P}(a^0)/\text{fin}$. Similarly we can find h^1 and h^2 which induce F on $\mathcal{P}(a^1)/\text{fin}$ and $\mathcal{P}(a^2)/\text{fin}$ respectively. By combining them and changing on a finite set if necessary we obtain the desired conclusion.

Let us now consider the general case, that the function φ is defined only on a dense G_δ set X . So there are open dense sets U_n , $n < \omega$ such that $X = \bigcap_{n < \omega} U_n$. We construct by induction an increasing sequence of integers $\langle m_k: k < \omega \rangle$ and a sequence of finite functions $\langle f_k: k < \omega \rangle$ such that $f_k: [m_k, m_{k+1}) \rightarrow 2$ and, for every $s \in 2^{m_k}$, $N_{s \cup f_k} \subseteq U_k$. Then let $f^\varepsilon = \bigcup\{f_k: k \equiv \varepsilon \pmod{2}\}$ and $a^\varepsilon = \bigcup\{[m_k, m_{k+1}): k \equiv \varepsilon \pmod{2}\}$ for $\varepsilon = 0, 1$. Then for every $x \subseteq a^\varepsilon$, $x \cup f^{1-\varepsilon} \in X$ so we can define $\varphi^\varepsilon: \mathcal{P}(a^\varepsilon) \rightarrow 2$ by $\varphi^\varepsilon(x) = \varphi(x \cup f^{1-\varepsilon}) \setminus \varphi(f^{1-\varepsilon})$. Clearly φ^ε is continuous for $\varepsilon = 0, 1$. So, by the special case of the lemma we get h^0, h^1 which induce F on $\mathcal{P}(a^0)$ and $\mathcal{P}(a^1)$ respectively. But then by combining them we conclude that F is trivial. $\square$

The above lemma clearly implies that if $F \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$ and there is a Baire choice function for F , then F is trivial. We also have the corresponding result for a Lebesgue measurable choice function.

PROPOSITION 3. *Let $F \in \text{Aut}(\mathcal{P}(\omega)/\text{fin})$. Suppose there is a set X of positive measure and a Lebesgue measurable function $\varphi: X \rightarrow 2^\omega$ such that $\forall x \in X$ $F[x] = [\varphi(x)]$. Then F is trivial.*

PROOF. By Luzin's theorem (see [8, p. 53]) there is a compact set C of positive measure such that $\varphi \upharpoonright C$ is continuous. Let $s \in 2^{<\omega}$ be such that $\mu(C \cap N_s) > \frac{1}{2}\mu(N_s)$. By changing φ if necessary we may assume s is a sequence of 0's of length $n < \omega$. Consider the addition modulo 2, i.e. $x + y$ is such that $\forall k < \omega$, $x(k) + y(k) \equiv z(k) \pmod 2$ for $x, y \in 2^\omega$, $z \in 2^\omega$.

Claim. $N_s \subseteq C + C = \{x + y : x, y \in C\}$.

PROOF. Fix $z \in N_s$. Then $z + (C \cap N_s) \subseteq N_s$ and $\mu(z + (C \cap N_s)) > \frac{1}{2}\mu(N_s)$. So, $(z + C) \cap C \neq \emptyset$. Pick $x, y \in C$ such that $z + x = y$. But then also $z = x + y$.

Now, consider the lexicographical ordering of 2^ω . Define

$$h(z) = \min((z + C) \cap C)$$

and $g(z) = z + h(z)$ for $z \in N_s$. Finally, let $\psi(z) = \varphi(h(z)) + \varphi(g(z))$. Then ψ is a Baire function, $\psi: N_s \rightarrow 2^\omega$ and $\forall z \in N_s$, $F[z] = [\psi(z)]$. This is sufficient to conclude, by virtue of the previous lemma, that F is trivial. $\square$

COROLLARY 4. *Every Borel automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial.*

PROOF. Let F be a Borel automorphism. So $\text{Gr}(F)$ is a Borel subset of $2^\omega \times 2^\omega$ with countable sections. But such sets can be uniformized by Borel sets (see Moschovakis [6, p. 258]). Hence there is a function $\varphi: 2^\omega \rightarrow 2^\omega$ such that $\langle x, \varphi(x) \rangle \in \text{Gr}(F)$ for every $x \in 2^\omega$ and $\text{Gr}(\varphi)$ is Borel. But it is well known that such φ is continuous on a dense G_δ set, so the result follows from Lemma 2. $\square$

Clearly, the above result is true for analytic automorphisms, too, since they are in fact Borel.

COROLLARY 5. *Axiom of Determinacy implies that every automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial.*

PROOF. AD implies that every $G \subseteq 2^\omega \times 2^\omega$ with nonempty sections can be uniformized by a continuous function on a dense G_δ set. This is a well-known fact, first proved by Solovay, but let us sketch the argument for the interested reader. Consider the following infinite game played by two players, I and II.

I	σ_0	σ_1	σ_2	$\dots$
II	τ_0	τ_1	τ_2	$\dots$
	n_0	n_1	n_2	

I plays finite partial functions σ_i (i.e. elements of $2^{<\omega}$) while II plays finite partial functions τ_i and integers n_i . In the end they produce two sequences $x = \sigma_0 \cap \tau_0 \cap \sigma_1 \cap \tau_1 \cap \dots$ and $y = \langle n_0, n_1, \dots \rangle$. We declare that II wins iff $\langle x, y \rangle \in G$.

A winning strategy for II easily gives rise to a uniformizing continuous function on a dense G_δ set. By AD we will be done once we prove the following.

Claim. I does not have a winning strategy.

PROOF. Fix a strategy S for I. Enumerate $2^{<\omega}$ as $\{s_i : i < \omega\}$ in such a way that $s_i \subseteq s_j$ implies $i \leq j$. We build by induction an increasing sequence of elements of $2^{<\omega}$, $\{q_i : i < \omega\}$ and a sequence of partial plays $\{p_i : i < \omega\}$ such that

(i) p_i is a partial play according to S of the form

$$\begin{array}{ccccccc} \sigma_0^i & & \sigma_1^i & & \cdots & & \sigma_k^i \\ & \tau_0^i & & \tau_1^i & & \cdots & \tau_{k-1}^i \\ & s_i(0) & & s_i(1) & & & s_i(k-1) \end{array}$$

where $k = \text{length}(s_i)$ and such that $\sigma_0^i \cap \tau_0^i \cap \cdots \cap \sigma_k^i = q_i$.

(ii) If s_j extends s_i then p_j extends p_i .

We proceed by induction. Suppose we have built p_i and q_i for $i \leq n$. We construct p_{n+1} and q_{n+1} . First, look at s_{n+1} . Let $\text{length}(s_{n+1}) = k + 1$ and let $i \leq n$ be such that $s_i = s_{n+1} \upharpoonright k$. Consider the partial play p_i . Let $\tau_j^{n+1} = \tau_j^i$ for $j \leq k - 1$ and $\sigma_j^{n+1} = \sigma_j^i$ for $j \leq k$. Then let τ_k^{n+1} be such that $q_i \cap \tau_k^{n+1} = q_n$ and $\sigma_{k+1}^{n+1} = S(p_i \cap \langle \tau_k^{n+1}, s_{n+1}(k) \rangle)$. Finally put $p_{n+1} = p_i \cap \langle \tau_k^{n+1}, s_{n+1}(k) \rangle \cap \sigma_{k+1}^{n+1}$ and $q_{n+1} = q_n \cap \sigma_{k+1}^{n+1}$. This finishes the inductive construction of the p_i 's and q_i 's.

Now, let $x = \bigcup \{q_i : i < \omega\}$. It is clear that for every $y \in 2^\omega$ there is a run of the game p in which I uses his strategy S

$$\begin{array}{ccccccc} \sigma_0 & & \sigma_1 & & \cdots & \sigma_n & \cdots \\ & \tau_0 & & \tau_1 & & \cdots & \tau_n & \cdots \\ & y(0) & & y(1) & & & y(n) & \end{array}$$

such that $\sigma_0 \cap \tau_0 \cap \sigma_1 \cap \tau_1 \cap \cdots \cap \sigma_n \cap \tau_n \cap \cdots = x$. Namely, if $y \upharpoonright n = s_{i_n}$ we know that p_{i_n} extends p_{i_m} for $n < m$, so we can take $p = \bigcup \{p_{i_n} : n < \omega\}$. If y is chosen such that $\langle x, y \rangle \in G$, then II wins this run of the game thus showing that S is not a winning strategy for I. $\square$

Our next theorem shows that Borel cannot be replaced by projective (in ZFC) in Corollary 2. However we do not know what happens with automorphisms which are between Σ_1^1 and Δ_2^1 in complexity.

THEOREM 6. *The Axiom of Constructibility implies that there is a nontrivial Δ_2^1 automorphism of $\mathcal{P}(\omega)/\text{fin}$.*

PROOF. This can be proved by simply modifying Baumgartner's argument mentioned at the beginning of this paper, constructing the sequences $\{a_\alpha : \alpha < \omega_1\}$ and $\{f_\alpha : \alpha < \omega_1\}$ in the canonical way using a fixed good Δ_2^1 -well ordering of $\mathcal{P}(\omega)$ (see Devlin [1, p. 70]). The details are straightforward and are left to the reader. $\square$

We finally show that it is consistent with CH that all projective, in fact, ODR automorphisms are trivial. We use the well-known model of Solovay [1], see also the presentation in Jech [2, p. 537].

Let κ be an infinite cardinal. The Levy collapse $\mathcal{L}_\kappa$ is the following poset. Conditions are finite functions p , such that $\text{dom}(p) \subseteq \kappa \times \omega$ and $p(\alpha, n) \in \alpha$ for $(\alpha, n) \in \text{dom}(p)$. Order is reverse inclusion. Assuming that κ is inaccessible, Solovay [11] showed that the following holds in $V^{\mathcal{L}_\kappa}$.

Fact. Every ODR subset of $2^\omega \times 2^\omega$ with nonempty sections can be uniformized on a dense G_δ set by a continuous function.

By Lemma 2 this immediately yields the following.

THEOREM 7. *If the theory “ZF + $\exists$ inaccessible cardinal” is consistent, then so is the theory “ZFC + CH + every ODR automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial”.*

REMARKS. (1) Shelah [10] has shown that the above uniformization result can be proved consistent without using an inaccessible cardinal. So, for Theorem 7 we need only assume that ZF is consistent.

(2) It follows from a recent result of Foreman, Magidor, Shelah and Woodin that if there is a supercompact cardinal (in fact much less) every automorphism of $\mathcal{P}(\omega)/\text{fin}$ which is in $L[\mathcal{P}(\omega)]$ is trivial.

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