TIGHTNESS IN PRODUCT SPACES

U. N. B. DISSANAYAKE AND S. W. WILLARD¹

ABSTRACT. A product $\prod X_i$ of topological spaces X_i , $i \in I$, will be said to *preserve tightness* if

$$\partial \left(\prod X_i \right) \leq |I| \cdot \sup \{ \partial(X_i) \mid i \in I \}$$

where $\partial(X)$ denotes the tightness of X .

We show $\prod X_i$ preserves tightness whenever each finite subproduct does. It is further shown that this is the case whenever each X_i is a locally compact T_2 -space, and whenever each X_i is a locally Lindelöf T_3 P -space, extending 5.9 in [J].

Preliminaries. Let m and n be infinite cardinals, $m \geq n$. A space X is said to be *m - n compact* iff every open cover $\mathcal{U}$ of X of cardinality $\leq m$ has a subcover of cardinality $< n$. If every open cover of X admits a subcover of cardinality $< n$, we call X *∞ - n compact*. These spaces have been considered in [G, N and V]. Our notation follows that in [N].

A filter $\mathcal{F}$ on X is an *m - n filter* provided it has a filter base $\mathcal{F}_0$ with cardinality $\leq m$ and the intersection of any subfamily of cardinality $< n$ is nonvoid (i.e. $\mathcal{F}$ has the *$< n$ -intersection property*).

A space X is *$< n$ -discrete* provided every intersection of fewer than n open sets is open.

We quote the following standard results for future reference.

1.1 THEOREM (GAL [G, THEOREM 1.1]). *A space X is m - n compact iff every m - n filter on X has a cluster point.*

1.2 THEOREM (KUNEN). *Every ∞ - n compact, $< n$ -discrete T_3 -space is paracompact.*

1.3 THEOREM (VAUGHAN [V, THEOREM 3.2]). *Let n be a regular cardinal. If X is ∞ - n compact and $< n$ -discrete for $i = 1, 2, \dots$, then $\prod_{i=1}^{\infty} X_i$ is ∞ - n compact.*

The following result will also be needed in the sequel. It can easily be derived as a consequence of Theorem 2.2 in [N]; a direct proof is not difficult and will be provided here.

1.4 THEOREM. *If X is $< n$ -discrete and has character $\leq m$, and Y is m - n compact, the projection $\pi_X: X \times Y \rightarrow X$ is closed.*

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PROOF. Let $F \subset X \times Y$ be closed, $x \in \overline{\pi_x(F)}$. Let ν_X be a neighbourhood system at x in X with $|\nu_X| \leq m$. Set

$$\mathcal{G} = \{\pi_X^{-1}(V) \cap F \mid V \in \nu_X\},$$

and let $\pi_Y(\mathcal{G}) = \{\pi_Y(G) \mid G \in \mathcal{G}\}$. Then easily $\pi_Y(\mathcal{G})$ is a base for an m - n filter on Y , and hence by 1.1, we can find some $y \in \{\bigcap\{\overline{\pi_Y(G)} \mid G \in \mathcal{G}\}\}$.

Now if V and W are neighbourhoods of x and y , W meets $\pi_Y[\pi_X^{-1}(V) \cap F]$ and hence $\pi_Y^{-1}(W) \cap \pi_X^{-1}(V) \cap F \neq \emptyset$. Thus every (basic) neighbourhood $V \times W$ of (x, y) in $X \times Y$ meets F , so $(x, y) \in \overline{F} = F$, and hence $x \in \pi_X(F)$.

Finally, we introduce some new terminology. A space X will be called *locally ∞ - n compact* if each $x \in X$ has a base of ∞ - n compact neighbourhoods. We will call X a *GC(n, i)-space*, for $i = 1, 2, 3, 4$, if X is an ∞ - n compact, $< n$ -discrete T_i space, and X will be a *GLC(n, i)-space*, for $i = 1, 2, 3, 4$, if X is a locally ∞ - n compact, $< n$ -discrete T_i space.

2. Tightness in small products. We denote the tightness (cf. Juhasz [J, 1.16]) of a topological space X by $\partial(X)$. Recall that a subset F of X is κ -closed, where κ is an infinite cardinal iff whenever $S \subset F$ and $|S| \leq \kappa$ then $\overline{S} \subset F$.

The following facts are standard and will be used in the sequel without specific reference:

- (a) Every closed set in X is κ -closed.
- (b) Any intersection of κ -closed sets is κ -closed.
- (c) If f is a closed map of X onto Y and $F \subset X$ is κ -closed in X , then $f(F)$ is κ -closed in Y .
- (d) $\partial(X) \leq \kappa$ iff every κ -closed set in X is closed.

The results of this section will stem from the following basic result. The proof mimics in detail the proof given in [J, p. 113] for the case $n = \aleph_0$.

2.1 THEOREM. *If X is a $< n$ -discrete T_1 -space and Y is an ∞ - n compact regular space, then*

$$\partial(X \times Y) \leq \partial(X) \cdot \partial(Y).$$

PROOF. Let $\kappa = \partial(X) \cdot \partial(Y)$ and suppose $H \subset X \times Y$ is κ -closed. It suffices to show H is closed.

Let $(p, q) \in H$. If $T = H \cap (\{p\} \times Y)$, then T is κ -closed and hence closed in $\{p\} \times Y$, and hence closed in $X \times Y$. We need only show $q \in \pi_Y(T)$.

Suppose $q \notin \pi_Y(T)$. Since $\pi_Y|_{\{p\} \times Y}$ is a homeomorphism, $\pi_Y(T)$ is closed in Y . Let V be a closed neighbourhood of q such that $V \cap \pi_Y(T) = \emptyset$. Note that since $X \times V$ is a neighbourhood of (p, q) and $(p, q) \in \overline{H}$, we have $(p, q) \in \overline{(X \times V) \cap H}$. But $(X \times V) \cap H$ is a closed subset of H , and hence κ -closed in $X \times Y$. Since Y is ∞ - n compact and X is $< n$ -discrete, π_X is closed (by 1.4). Thus $\pi_X[(X \times V) \cap H]$ is κ -closed and hence closed in X . Then, since π_X is continuous,

$$p \in \pi_X[\overline{(X \times V) \cap H}] \subset \overline{\pi_X[(X \times V) \cap H]} = \pi_X[(X \times V) \cap H].$$

So for some $v \in V$, $(p, v) \in H$. But then $v \in \pi_Y(T) \cap V$, a contradiction.

2.2 THEOREM. *If X is a $< n$ -discrete T_1 and Y is locally ∞ - n compact T_3 , then $\partial(X \times Y) \leq \partial(X) \partial(Y)$.*

PROOF. Let $K = \partial(X) \cdot \partial(Y)$ and let $H \subset X \times Y$ be κ -closed. Choose $(p, q) \in \overline{H}$. Let V be any closed ∞ - n compact neighbourhood of q . Then $X \times V$ is closed in

$X \times Y$ and $\partial(X \times V) \leq \kappa$ by 2.1. But $H \cap (X \times V)$ is κ -closed in $X \times V$, and thus closed in $X \times V$, and $(p, q) \in \text{Cl}_{X \times V} H \cap (X \times V) = H \cap (X \times V)$. Thus $(p, q) \in H$.

2.3 COROLLARY. If $X_1, X_2, \dots, X_n$ are $\text{GLC}(n, 3)$ spaces, then

$$\partial \left(\prod_{i=1}^n X_i \right) \leq \partial(X_1) \cdot \dots \cdot \partial(X_n).$$

PROOF. The proof is by induction, noting that

$$X_1 \times \dots \times X_k = (X_1 \times \dots \times X_{k-1}) \times X_k$$

and that $X_1 \times \dots \times X_{k-1}$ is $< n$ -discrete and T_1 , while X_k is locally ∞ - n compact and T_3 , so that

$$\partial(X_1 \times \dots \times X_k) \leq \partial(X_1 \times \dots \times X_{k-1}) \cdot \partial(X_k)$$

by 2.2.

Thus, finite products of locally compact T_2 spaces ($= \text{GLC}(\aleph_0, 2)$) preserve tightness, as do finite products of locally Lindelöf T_3 P -spaces ($= \text{GLC}(\aleph_1, 3)$), and so on.

Infinite products will be considered in the next section; first an example, showing that Theorem 2.2 cannot be improved for T_3 spaces.

2.4 EXAMPLE. In [T], an example is given of a Lindelöf space Y and an ∞ - c^+ compact space X , each sequential and therefore of countable tightness, whose product has uncountable tightness. Since only an outline of proof is included in the reference, we will develop the example fully here.

The space Y , the *countable fan*, is the quotient obtained by identifying to a single point all nonisolated points in the disjoint union of denumerably many copies of the converging sequence, while X , the *c-fan*, is obtained by applying the same process to c copies of the converging sequence. Each is a quotient of a metric space, hence sequential, hence of countable tightness. To show that $X \times Y$ fails to have countable tightness, we will employ a more notational approach to X and Y .

Let X and Y be the following subsets of $R \times R$:

$$X = \{(0, 0)\} \cup \{(r, n) | r \in R^+, n = 1, 2, \dots\},$$

$$Y = \{(0, 0)\} \cup \{(m, n) | m = 1, 2, \dots, n = 1, 2, \dots\}.$$

All points in X and Y will be isolated except the points $(0, 0)$. A typical basic neighbourhood B of $(0, 0)$ in X will be obtained by choosing a positive integer m_r for each $r \in R^+$ and setting

$$B = \{(0, 0)\} \cup \{(r, n) | n > m_r\},$$

while a typical basic neighbourhood C of $(0, 0)$ in Y will be obtained by choosing a positive integer n_m for each $m = 1, 2, \dots$ and setting

$$C = \{(0, 0)\} \cup \{(m, n) | n > n_m\}.$$

We claim $X \times Y$ does not have countable tightness; we will, in fact, produce a subset S of $[X - \{(0, 0)\}] \times [Y - \{(0, 0)\}]$ such that $(0, 0, 0, 0) \in \bar{S}$ in $X \times Y$, but $(0, 0, 0, 0)$ is not in the closure of any countable subset of S .

Note that the basic neighbourhood system $\mathcal{C}$ at $(0, 0)$ in Y has cardinality c and let f be any one-one map of $\mathcal{C}$ onto R^+ , denoting $f(C)$ by r_C for each $C \in \mathcal{C}$.

For $C \in \mathcal{C}$, select a sequence $X_{C1}, X_{C2}, \dots$ in $X \times Y$ by choosing n_{Cm} for each $m = 1, 2, \dots$ so that $(m, n_{Cm}) \in C - \{0, 0\}$ and setting

$$X_{Cm} = (r_C, m, m, n_{Cm}).$$

Now set $S = \{X_{Cm} | C \in \mathcal{C}, m = 1, 2, \dots\}$.

We first show $(0, 0, 0, 0) \in \bar{S}$ in $X \times Y$. Let $B \times C$ be a (basic) neighbourhood of $(0, 0, 0, 0)$ in $X \times Y$. Then B includes for some choice(s) of m , a point of the form (r_C, m) and then $X_{Cm} = (r_C, m, m, n_{Cm})$ belongs to $\{(r_C, m)\} \times (C - \{(0, 0)\})$ and hence to $B \times C$.

Now we claim $(0, 0, 0, 0) \notin \bar{M}$ if $M \subset S$ is countable. The first coordinates of points of M occur, let us say, as $r_{C_1}, r_{C_2}, \dots$. We will construct a neighbourhood of $(0, 0, 0, 0)$ in $X \times Y$ which includes no such point of S . Choose a (basic) neighbourhood B of $(0, 0)$ in X by setting $m_{r_{C_n}} = n$ for each $n = 1, 2, \dots$ (and letting m_r be any positive integer where $r \neq r_{C_n}$ for some n). Choose a (basic) neighbourhood C of $(0, 0)$ in Y by setting

$$n_l > \max\{n_{C_1l}, n_{C_2l}, \dots, n_{C_l l}\}$$

for each $l = 1, 2, \dots$. Now we claim $B \times C$ contains no point in S with first coordinate r_{C_k} for any k , that is, no point of the form $X_{C_k l} = (r_{C_k}, l, l, n_{C_k l})$. For if $(r_{C_k}, l) \in B$, then $l > k$, and hence $n_l > n_{C_k l}$ and $(l, n_{C_k l}) \notin C$. Thus $B \times C$ contains no point of M .

3. Tightness in large products. If $\mathcal{R} = \{R_i | i \in I\}$ is any collection of topological spaces, we define

$$\partial_I(\mathcal{R}) = \sup\{\partial(R_i) | i \in I\}.$$

We will say the product $R = \prod_{i \in I} R_i$ *preserves tightness* iff

$$\partial\left(\prod_{i \in I} R_i\right) \leq |I| \cdot \partial_I(\mathcal{R}).$$

For finitely many spaces $X_1, \dots, X_n$ this reduces to the assertion that

$$\partial\left(\prod_{i=1}^n X_i\right) \leq \partial(X_1) \cdot \dots \cdot \partial(X_n),$$

and thus the results of the previous section.

It will be convenient to introduce some notation. Let $R = \prod_{i \in I} R_i$ and suppose $J \subset I$. The subproduct $\prod_{i \in J} R_i$ of R will be denoted R_J , and the projection of R onto R_J will be denoted π_J . For $a \in R$ and $A \subset R$ we will denote by a_J and A_J the images $\pi_J(a)$ and $\pi_J(A)$, respectively.

Our main result is a generalization of a part of Theorem 5.9 in [J] and is proved in similar fashion. Since it reduces the question of preservation of tightness in a large product to consideration of its small subproducts, our concrete results will be corollaries to results of the previous section.

3.1 THEOREM. *If each finite subproduct of a product $R = \prod_{i \in I} R_i$ preserves tightness, then R preserves tightness.*

PROOF. Let $\mathcal{R} = \{R_i | i \in I\}$ and set $\kappa = |I| \cdot \partial_I(\mathcal{R})$. Let $I^{(F)} = \{J \subset I | |J| < \aleph_0\}$.

Suppose $A \subset R$ is κ -closed, and $a \in \bar{A}$. Then for $J \in I^{(F)}$, a_J belongs to $\pi_J(\bar{A}) \subset \bar{A}_J$. Since $\partial(R_J) \leq \kappa$, we can find $B_J \subset A_J$ with $|B_J| \leq \kappa$ such that $a_J \in \bar{B}_J$.

For each $b \in B_J$, choose $x_b \in A$ so that $\pi_J(x_b) = b$, and set $C_J = \{x_b | b \in B_J\}$. Clearly $|C_J| = |B_J| \leq \kappa$ and hence if $C = \bigcup \{C_J | J \in I^{(F)}\}$, then $C \subset A$ and $|C| \leq \kappa \cdot |I| = \kappa$.

But $a \in \bar{C}$. For if $U = U_J \times \prod_{i \in J} R_i$ is a basic open neighbourhood of a , then since $\pi_J(a) \in \bar{B}_J = \overline{\pi_J(C_J)}$, we have $U_J \cap \pi_J(C_J) \neq \emptyset$, and thus $U \cap C \neq \emptyset$, whence $U \cap C \neq \emptyset$.

Thus $a \in A$, and A is closed.

3.2 COROLLARY. *If R_i is a $\text{GLC}(n, 3)$ space for each $i \in I$, then $\prod R_i$ preserves tightness.*

PROOF. Apply 3.1 and 2.3.

In [J, 5.9] it is shown that any product of compact T_2 spaces preserves tightness. The first of the various special cases of 3.2 generalizes this result.

3.3 COROLLARY. *If R_i is locally compact and T_2 , for each $i \in I$, then $\prod R_i$ preserves tightness.*

PROOF. Locally compact, $T_2 \equiv \text{GLC}(\aleph_0, 2)$.

3.4 COROLLARY. *If R_i is a locally Lindelöf, T_3 P -space, for each $i \in I$, then $\prod R_i$ preserves tightness.*

PROOF. Locally Lindelöf, T_3 P -space $\equiv \text{GLC}(\aleph_1, 3)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1