ABSTRACT NASH MANIFOLDS

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ABSTRACT. We prove that any abstract noncompact Nash manifold is C^∞ diffeomorphic to the interior of some compact C^∞ manifold with boundary, and conversely, that such an interior or a compact C^∞ manifold admits infinitely many abstract Nash manifold structures. The last result is a generalization of [2], where the case of a torus is proved.

1. Introduction. A continuous map of semialgebraic sets is called *semialgebraic* if the graph is semialgebraic. Let Γ denote the pseudogroup of all semialgebraic analytic diffeomorphisms between semialgebraic open subsets of a Euclidean space. An *abstract Nash manifold* is a manifold with Γ -structure and with a finite system of coordinate neighborhoods. An analytic map $f: M_1 \rightarrow M_2$ of abstract Nash manifolds is called a *Nash map* if for each point $x \in M_1$, there exist coordinate neighborhoods $h_1: U_1 \rightarrow \mathbf{R}^n$ and $h_2: U_2 \rightarrow \mathbf{R}^m$ of $x, f(x)$ in M_1, M_2 respectively, such that $h_2 \circ f \circ h_1^{-1}: h_1(U_1 \cap f^{-1}(U_2)) \rightarrow \mathbf{R}^m$ is semialgebraic. If an abstract Nash manifold has a Nash imbedding in some Euclidean space, we call it an *affine Nash manifold*. We can define the above concepts by terms familiar to algebraists (see [1] and [2]). In this paper, manifolds do not have boundary unless otherwise specified.

Affine Nash manifold structures on C^∞ manifolds were studied and clarified by Nash [5] and the author [7]. To be precise, a compact C^∞ manifold admits the unique compatible affine Nash manifold structure [5], and this holds true even if the manifold has boundary [7]. For each noncompact affine Nash manifold M , there exists uniquely up to Nash diffeomorphism a compact affine Nash manifold with boundary whose interior is Nash diffeomorphic to M [7]; in a word, M admits a unique compactification.

We know little about abstract Nash manifolds. On S^1 , Mazur [6] showed a nonaffine abstract Nash manifold structure. Moreover, Chillingworth, Hubbard, and Mazur [2] proved that any torus admits infinitely many distinct abstract Nash manifold structures. We want to generalize these results. A C^∞ manifold is called *compactifiable* if it is C^∞ diffeomorphic to the interior of some compact C^∞ manifold with boundary.

THEOREM 1. *Each abstract Nash manifold is compact or compactifiable as a C^∞ manifold.*

THEOREM 2. *Any compact or compactifiable C^∞ manifold admits infinitely many compatible nonaffine abstract Nash manifold structures.*

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In §2 we introduce the concept of a C^r Nash manifold for $0 < r < \infty$ and prove that any abstract C^r Nash manifold is affine. As a consequence, we obtain Theorem 1. See [8] for detailed properties of a C^0 Nash manifold structure.

2. C^r Nash manifold. Let $0 < r < \infty$. Replacing “analytic” in §1 by “ C^r ”, we define an abstract C^r Nash manifold.

REMARK 2.1. If we define a C^r Nash map in the same way as in §1, then a Nash map of Euclidean spaces is locally semialgebraic but not necessarily semialgebraic. For example, $f(x) = (x - [x] - 1)^{r+1}(x - [x])^{r+1}$, where $[\]$ denotes the Gauss symbol. Such a function holds no interesting global property because it is not “of finite type”.

REMARK 2.2. Let $f: M_1 \rightarrow M_2$ be a Nash map from an abstract Nash manifold to an affine one. Then, for each pair $h_1: U_1 \rightarrow \mathbf{R}^n$ and $h_2: U_2 \rightarrow \mathbf{R}^m$ of coordinate neighborhoods of M_1 and M_2 , $h_1(U_1 \cap f^{-1}(U_2)) \subset \mathbf{R}^n$ and $h_2 \circ f \circ h_1^{-1}: h_1(U_1 \cap f^{-1}(U_2)) \rightarrow \mathbf{R}^m$ are semialgebraic. The proof below of Theorem 2 shows that it is not necessarily the case for general M_2 .

PROOF OF REMARK 2.2. First consider the case $M_1 = U_1 \subset \mathbf{R}^n$, $M_2 = U_2 \subset \mathbf{R}^m$, and $h_1 = h_2 = \text{ident}$. Because the graph of f is closed in $U_1 \times \mathbf{R}^m$ and consists of a finite number of connected components, it suffices to prove the following. Given an open semialgebraic set $W \subset \mathbf{R}^l$ and a closed subset $X \subset W$, assume that X is a connected analytic manifold and that each $x \in X$ has a semialgebraic neighborhood in X (i.e., X is locally semialgebraic). Then X is semialgebraic. Indeed, let V be a semialgebraic open subset of X . Let $g_1, \dots, g_k$ be generators of the ideal of V in the polynomial ring on $\mathbf{R}^l$. Then they vanish on X since X is a connected analytic manifold. Let Y be the set of common zeros and Z be the set of $y \in Y$ where some neighborhoods of y in Y are analytic manifolds of dimension $= \dim X$. Then $Z \cap W$ is a semialgebraic analytic manifold and hence consists of a finite number of connected components, and $Y - Z$ is closed in $\mathbf{R}^l$ and is of dimension $< \dim X$, e.g. [2]. Because $X \cap Z$ is an analytic manifold that is closed in $W - (Y - Z)$ and contained in $Z \cap W$, it is a union of some connected components of $Z \cap W$. Hence, $X \cap Z$ is semialgebraic [2]. It follows from $\dim(Y - Z) < \dim X$ that the closure of $X \cap Z$ in W is X , which proves that X is semialgebraic.

The case $M_2 = \mathbf{R}^m$ follows immediately from the proof above. For the general case let $\phi: M_2 \rightarrow \mathbf{R}^l$ be a Nash imbedding. Then $\phi \circ f: M_1 \rightarrow \mathbf{R}^l$ is a Nash map. Hence by the second case, $\phi \circ f \circ h_1^{-1}: h_1(U_1) \rightarrow \mathbf{R}^l$ is semialgebraic. The second case also implies that $\phi \circ h_2^{-1}: h_2(U_2) \rightarrow \mathbf{R}^l$ is semialgebraic, which shows that $\phi(U_2) = \phi \circ h_2^{-1}(h_2(U_2)) \subset \mathbf{R}^l$ is semialgebraic. Remark 2.2 follows from these, because

$$(\phi \circ f \circ h_1^{-1})^{-1}(\phi(U_2)) = h_1(U_1 \cap f^{-1}(U_2))$$

and because the graph of $h_2 \circ f \circ h_1^{-1}: h_1(U_1 \cap f^{-1}(U_2)) \rightarrow \mathbf{R}^m$ is the image of

$$\{(x, y, z) \in h_1(U_1) \times h_2(U_2) \times \mathbf{R}^l \mid z = \phi \circ f \circ h_1^{-1}(x) = \phi \circ h_2^{-1}(y)\}$$

under the projection $h_1(U_1) \times h_2(U_2) \times \mathbf{R}^l \rightarrow h_1(U_1) \times h_2(U_2)$.

Let $f: M_1 \rightarrow M_2$ be a C^r map of abstract C^r Nash manifolds. If $h_1(U_1 \cap f^{-1}(U_2))$ and $h_2 \circ f \circ h_1^{-1}: h_1(U_1 \cap f^{-1}(U_2)) \rightarrow \mathbf{R}^m$ are semialgebraic for each pair $h_1: U_1 \rightarrow \mathbf{R}^n$ and $h_2: U_2 \rightarrow \mathbf{R}^m$ of coordinate neighborhoods of M_1 and M_2 , then we call f a C^r Nash map. An affine C^r Nash manifold is an abstract

C^r Nash manifold with a C^r Nash imbedding in some Euclidean space. A property of C^r Nash manifolds completely different from Nash manifolds is the following.

THEOREM 2.3. *An abstract C^r Nash manifold is affine.*

PROOF. Let M be an abstract C^r Nash manifold, and $h_i: U_i \rightarrow \mathbf{R}^n, i = 1, \dots, k$, be a system of coordinate neighborhoods of M . By a theorem of Mostowski [4], there are semialgebraic functions $\phi_i, i = 1, \dots, k$, on $\mathbf{R}^n$ such that $\phi_i^{-1}(0) = \overline{h_i(U_i)} - h_i(U_i)$ and $\phi_i|_{h_i(U_i)}$ are Nash functions. For each i , set $h'_i = (h_i, 1/\phi_i \circ h_i)$. Then h'_i is a proper C^r Nash imbedding of U_i in $\mathbf{R}^{n+1}$. As $\mathbf{R}^{n+1}$ is Nash diffeomorphic to S^{n+1} —a point by the stereographic projection, h'_i induces a C^r Nash imbedding $h''_i: U_i \rightarrow \mathbf{R}^{n+2}$ such that $h''_i(U_i)$ is bounded and that $\overline{h''_i(U_i)} - h''_i(U_i)$ is a point. Assume the point = 0. Set

$$\psi(x_1, \dots, x_{n+2}) = \sum_{j=1}^{n+2} x_j^{2l} \times (x_1, \dots, x_{n+2}), \quad g_i = \psi \circ h''_i$$

for a sufficiently large integer l . Then g_i holds the same properties as h''_i ; moreover, Łojasiewicz' inequality says that we can extend g_i to M as a C^r Nash map by defining $g_i = 0$ on $M - U_i$. Hence, $g: \prod_{i=1}^k g_i: M \rightarrow \mathbf{R}^{k(n+2)}$ is a C^r Nash imbedding.

THEOREM 2.4. *A C^r Nash manifold is compact or compactifiable as a C^r manifold.*

PROOF. Let M be a noncompact C^r Nash manifold. By the proof of Theorem 2.3 there is a bounded C^r Nash imbedding $g: M \rightarrow \mathbf{R}^m$ such that $\overline{g(M)} - g(M)$ is a point, say 0. Hence, we can assume that M is a bounded subset of $\mathbf{R}^n$ satisfying $\overline{M} - M = 0$. We note that $\overline{M}$ is semialgebraic. Recall the well-known fact (e.g. [2]) that a semialgebraic set admits a Whitney stratification, from which it follows that the tangent space $T_x M$ is not perpendicular to the vector $0x$, for $x \in M$ near 0. Let v_x be the orthogonal projection image of $x0$ to $T_x M$. Then v is a C^{r-1} nonsingular vector field on M around 0. Assume $r \geq 2$. Then, integrating v we obtain a C^{r-1} diffeomorphism from $M \cap \{|x| \leq \varepsilon\}$ to $(M \cap \{|x| = \varepsilon\}) \times (0, 1]$ for small $\varepsilon > 0$. Here, $M \cap \{|x| = \varepsilon\}$ is a compact C^r manifold, and hence we can approximate the C^{r-1} diffeomorphism by a C^r one. Therefore, M is compactifiable. If $r = 1$, consider a close approximation M' of M by a C^∞ manifold in the Whitney C^1 topology such that M' and M are C^1 diffeomorphic and M' remains to satisfy the conditions on M (i.e., $\overline{M'} - M' = 0$ and $(M', 0)$ satisfies the Whitney condition). Then, by the above argument, M' is compactifiable as a C^∞ manifold. Hence, M is compactifiable as a C^1 manifold. Thus, the theorem is proved.

PROOF OF THEOREM 1. Trivial by Theorem 2.4.

3. Proof of Theorem 2. Let M be a compact or compactifiable C^∞ manifold. We will construct nonaffine abstract Nash manifolds M_c parametrized by $0 < c < 1$ such that

(3.1) M_c are diffeomorphic to M , and

(3.2) M_c and $M_{c'}$ are not Nash diffeomorphic if $\log c / \log c'$ is irrational. Let c be fixed, and set

$$a = 2^{5/2}(1+c)^2/(1-c)^2, \\ d = 2 + 2^{3/2}a - (a + 2^{1/2})(-a + (a^2 + 2^{5/2}a)^{1/2}).$$

We easily see that $2^{5/2} < a$ and $1 < d < 2$.

I. First consider the case $M = \mathbf{R}$. Set

$$N_1 = (-\infty, d), \quad N_2 = (0, \infty), \quad N_3 = (0, 1).$$

Let f be a Nash function on N_3 satisfying

$$2^{1/2}(x + f(x)) = (x - f(x))^2/a.$$

If we rotate the graph of f with center at 0, then it comes to the graph of $y = x^2/a$. From this, and from the inequality $2^{5/2} < a$ it follows that f and its Nash extension $\tilde{f}$ to $[1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}]$ is well defined so that

$$\tilde{f}[1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}] = [1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}],$$

the derivative of $\tilde{f} < 0$, and $\tilde{f} \circ \tilde{f} = \text{ident}$. Define Nash maps $h_1: N_3 \rightarrow N_1$ and $h_2: N_3 \rightarrow N_2$ by

$$h_1(x) = x^2 + f^2(x) \quad \text{and} \quad h_2(x) = -x^2 + 2x.$$

Then there are Nash imbeddings such that $h_1(N_3) = (0, d)$, $h_2(N_3) = (0, 1)$. Note that h_1 can be extended to $\tilde{h}_1: [1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}] \rightarrow \mathbf{R}$ as a Nash function whose derivative vanishes only at 0 and satisfies $\tilde{h}_1(x) = \tilde{h}_1(\tilde{f}(x))$ because the derivative of $\tilde{f} < 0$ and $\tilde{f} \circ \tilde{f} = \text{ident}$. Consider the disjoint union of N_1 , N_2 , and N_3 , and an equivalence relation $x \sim h_1(x)$ and $x \sim h_2(x)$ on the union. Then the quotient topological space admits an abstract Nash manifold structure induced by those of N_1 , N_2 , and N_3 . Let M_c denote the abstract Nash manifold. Trivially (3.1).

Put $\overline{N}_3 = [0, 1]$ and consider it to contain N_3 . Let $\rho: \overline{N}_3 \rightarrow M_c$ be the natural map. To prove (3.2), we use the Nash map

$$\phi: (1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}) \rightarrow \rho(\overline{N}_3) \subset M_c.$$

Define ϕ on $(0, 1)$ so that $\rho^{-1} \circ \phi = \text{ident}$, and extend it as widely as possible. By the method of patching of N_2 and N_3 , first ϕ is extended to $(0, 2)$ so that $\rho^{-1} \circ \phi(x) = 2 - x$ for $x \in [1, 2)$, and 1 is the only and nondegenerate critical point. Secondly, the patching of N_1 and N_3 says that ϕ can be defined on $(\tilde{f}(1), 0]$

satisfying $\rho^{-1} \circ \phi(x) = \tilde{f}(x)$ for $x \in (\tilde{f}(1), 0)$, and its critical point is only 0 and nondegenerate. Repeating these extensions, we have

$$\rho^{-1} \circ \phi(x) = \begin{cases} \vdots & \\ \tilde{f}(2 - \tilde{f}(2 - x)) & \text{on } [2 - \tilde{f}(2), 2 - \tilde{f}(2 - \tilde{f}(1))], \\ 2 - \tilde{f}(2 - x) & \text{on } [2 - \tilde{f}(1), 2 - \tilde{f}(2)], \\ \tilde{f}(2 - x) & \text{on } [2, 2 - \tilde{f}(1)], \\ 2 - x & \text{on } [1, 2], \\ x & \text{on } [0, 1], \\ \tilde{f}(x) & \text{on } [\tilde{f}(1), 0], \\ 2 - \tilde{f}(x) & \text{on } [\tilde{f}(2), \tilde{f}(1)], \\ \tilde{f}(2 - \tilde{f}(x)) & \text{on } [\tilde{f}(2 - \tilde{f}(1)), \tilde{f}(2)], \\ \vdots & \end{cases}$$

It is seen that ϕ is finally defined on $(1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2})$, that for any $e \in \overline{N}_3$,

$$(\rho^{-1} \circ \phi)^{-1}(e) = \{\dots, 2 - \tilde{f}(2 - \tilde{f}(e)), 2 - \tilde{f}(2 - e), 2 - \tilde{f}(e), 2 - e, e, \tilde{f}(e), \tilde{f}(2 - e), \tilde{f}(2 - \tilde{f}(e)), \tilde{f}(2 - \tilde{f}(2 - e)), \dots\},$$

which is in order from the largest to the smallest, and that the set of critical points of ϕ is $(\rho^{-1} \circ \phi)^{-1}(0) \cup (\rho^{-1} \circ \phi)^{-1}(1)$ and consists of only nondegenerate ones. Place $(\rho^{-1} \circ \phi)^{-1}(e)$ in order as $\dots < e_i < e_{i+1} < \dots, i \in \mathbf{Z}$. Then we have

$$(3.3) \quad (1 + 2^{-1/4}a^{1/2} - e_{i+2})/(1 + 2^{-1/4}a^{1/2} - e_i)$$

and

$$(1 - 2^{-1/4}a^{1/2} - e_{i-2})/(1 - 2^{-1/4}a^{1/2} - e_{-i})$$

converge to c as $i \rightarrow \infty$. Because the map $x \rightarrow \tilde{f}(2 - x)$ has fixed points only at the end points of the interval, it repels from $1 + 2^{-1/4}a^{1/2}$ and attracts to $1 - 2^{-1/4}a^{1/2}$ and its derivative at the latter point is c . Hence,

$$(1 - 2^{-1/4}a^{1/2} - e_{-i-2})/(1 - 2^{-1/4}a^{1/2} - e_{-i})$$

approaches c since $e_{-i-2} = \tilde{f}(2 - e_{-i})$. A similar analysis of the map $x \rightarrow 2 - \tilde{f}(x)$ shows the other convergence result.

Hence, it follows from Remark 2.2 that M_c is not affine because a semialgebraic set consists of a finite number of connected components. We will prove (3.2) by reduction to absurdity. Assume a Nash diffeomorphism $g: M_c \rightarrow M_{c'}$ for $0 < c, c' < 1$, with irrational $\alpha = \log c' / \log c$. Put

$$a = 2^{5/2}(1 + c)^2/(1 - c)^2, \quad a' = 2^{5/2}(1 + c')/(1 - c')^2.$$

To distinguish the maps ρ to M_c and $M_{c'}$, we write them as $\rho_c: N_3 \rightarrow M_c$ and $\rho_{c'}: N_3 \rightarrow M_{c'}$. Note that the images of N_1 and N_2 , in M_c and $M_{c'}$, are the largest affine connected Nash submanifolds in M_c and $M_{c'}$ for the following reason. If it were not so, there would exist an affine connected Nash manifold which is contained in M_c or $M_{c'}$, and contains $\rho_c(N_3)$ or $\rho_{c'}(N_3)$ respectively. But that is impossible by the above argument that says M_c and $M_{c'}$ are not affine. Hence

we have $g(\rho_c\{0,1\}) = \rho_{c'}\{0,1\}$. Hence $\phi_{c'}$ and $g \circ \phi_c$ have the same image $= \rho_{c'}[0,1]$ and the same set of critical values $= \rho_{c'}\{0,1\}$. Furthermore, their values vibrate infinitely from $\rho_{c'}(0)$ to $\rho_{c'}(1)$ and from $\rho_{c'}(1)$ to $\rho_{c'}(0)$. Hence, for any pair $e_0 \in (1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2})$ and $e'_0 \in (1 - 2^{-1/4}a'^{1/2}, 1 + 2^{-1/4}a'^{1/2})$ with $\phi_{c'}(e'_0) = g \circ \phi_c(e_0)$, there exists a homeomorphism

$$\tau: (1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}) \rightarrow (1 - 2^{-1/4}a'^{1/2}, 1 + 2^{-1/4}a'^{1/2})$$

such that $\tau(e_0) = e'_0$ and $\phi_{c'} \circ \tau = g \circ \phi_c$ on $(1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2})$. It is clear that τ is locally semialgebraic and hence semialgebraic by Remark 2.2. Recall that all critical points of ϕ_c and $\phi_{c'}$ are nondegenerate, which proves that τ is of class C^∞ . Therefore, τ is a Nash diffeomorphism. Set

$$\begin{aligned} \rho_c^{-1} \circ \phi_c(e_0) &= e, & (\rho_c^{-1} \circ \phi_c)^{-1}(e) &= \{\cdots < e_i < e_{i+1} < \cdots\}, \\ \rho_{c'}^{-1} \circ \phi_{c'}(e'_0) &= e', & (\rho_{c'}^{-1} \circ \phi_{c'})^{-1}(e') &= \{\cdots < e'_i < e'_{i+1} < \cdots\}. \end{aligned}$$

Then τ satisfies $\tau(e_i) = e'_i$ for any $i \in \mathbf{Z}$, or $\tau(e_i) = e'_{-i}$ for any $i \in \mathbf{Z}$. For simplicity, suppose that the first equality holds and that e_i and e'_i converge to 0 as $i \rightarrow -\infty$ after translating $(1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2})$ and $(1 - 2^{-1/4}a'^{1/2}, 1 + 2^{-1/4}a'^{1/2})$. Then it follows from (3.3)

$$(3.4) \quad e_{-i-2}/e_{-i} \text{ and } e'_{-i-2}/e'_{-i} \text{ converge to } c \text{ and } c' \text{ respectively as } i \rightarrow \infty.$$

Let X denote the smallest algebraic set in $\mathbf{R}^2$ containing the graph of τ . That is of dimension 1 since τ is semialgebraic. Let $P(x, y)$ be a polynomial whose set of zeros is X . Then $P(e_i, e'_i) = 0$ for any $i \in \mathbf{Z}$. Let $P(x, y) = \sum_{i=1}^l \delta_i x^{\beta_i} y^{\gamma_i}$, $\delta_i \in \mathbf{R}$ and $\beta_i, \gamma_i \in \mathbf{N}$, be an expansion. Then we note that $\beta_i + \alpha\gamma_i \neq \beta_{i'} + \alpha\gamma_{i'}$ for $i \neq i'$, since α is irrational. Set $P_i(x, y) = x^{\beta_i} y^{\gamma_i}$. For any $n \in \mathbf{Z}$, let $D(n)$ denote the $l \times l$ -matrix whose (i, j) -component is $P_i(e_{-n-2j+1}, e'_{-n-2j+1})$. Then

$$(\delta_1, \dots, \delta_l)D(n) = (P(e_{-n-1}, e'_{-n-1}), \dots, P(e_{-n-2l+1}, e'_{-n-2l+1})) = 0.$$

Particularly, the determinant $|D(n)| = 0$. On the other hand we have

$$|D(n)| = \prod_{i=1}^l P_i(e_{-n-1}, e'_{-n-1}) |D'(n)|,$$

where $D'(n)$ is the $l \times l$ -matrix whose (i, j) -component is

$$P_i(e_{-n-2j+1}, e'_{-n-2j+1}) / P_i(e_{-n-1}, e'_{-n-1}).$$

Now (3.4) means that each (i, j) -component of $D'(n)$ converges to $(c^{\beta_i} c'^{\gamma_i})^{j-1} = c^{(\beta_i + \alpha\gamma_i)(j-1)}$ as $n \rightarrow \infty$. Hence $|D'(n)|$ converges to a Vandermonde's determinant which equals $\prod_{i < i'} (c^{\beta_{i'} + \alpha\gamma_{i'}} - c^{\beta_i + \alpha\gamma_i}) \neq 0$. Therefore $|D(n)| \neq 0$ for large n , which is a contradiction. Thus we have proved (3.2) and hence the case $M = \mathbf{R}$.

II. The case $M = \mathbf{R}^n$, $n > 1$. Let $E(\varepsilon)$, $\varepsilon > 0$, denote the solid open sphere in $\mathbf{R}^n$ of radius ε with center at 0, and let $p: \mathbf{R}^n - 0 \rightarrow S^{n-1} = \overline{E(1)} - E(1)$ be the projection such that $p^{-1}(x) \cup 0$ is a ray for any $x \in S^{n-1}$. We use the notations N_1, N_2 , etc. defined in I. Set

$$\theta(x) = |x| - 1, \quad N_1^n = E(1 + d), \quad N_2^n = \mathbf{R}^n - \overline{E(1)},$$

$$N_3^n = E(2) - \overline{E(1)}, \quad \overline{N_3^n} = \overline{E(2)} - E(1).$$

Then $q = (p, \theta): (N_1^n - 0, N_2^n, \overline{N_3^n}) \rightarrow S^{n-1} \times (N_1 - (-\infty, -1], N_2, \overline{N_3})$ is a Nash diffeomorphism. Set

$$h_1^n = q^{-1} \circ (\text{id} \times h_1) \circ q: N_3^n \rightarrow N_1^n, \quad h_2^n = q^{-1} \circ (\text{id} \times h_2) \circ q: N_3^n \rightarrow N_2^n.$$

Let M_c be the abstract Nash manifold constructed by the patching h_1^n and h_2^n . Then (3.1) is trivial.

Let $\rho_c^n: \overline{N_3^n} \rightarrow M_c$ be the natural map, $0 \in M_c$ be the point which corresponds to $0 \in N_1^n$, and $p_c: M_c - 0 \rightarrow S^{n-1}$ be the projection defined through q . For any $z \in S^{n-1}$, define a Nash map $\phi_{cz}^n: (1 - 2^{-1/4}a^{1/2}, 1 + 2^{-1/4}a^{1/2}) \rightarrow M_c$ by

$$\phi_{cz}^n(x) = \rho_c^n \circ q^{-1}(z \times \rho^{-1} \circ \phi_c(x)).$$

Then ϕ_{cz}^n holds the same properties as ϕ_c : critical points are all nondegenerate, the set of critical values is $\rho_c^n \circ q^{-1}(z \times \{0, 1\})$, the set of values is $\rho_c^n \circ q^{-1}(z \times [0, 1])$, and ϕ_{cz}^n vibrates infinitely. Let $0 < c, c' < 1$. Assume $g: M_c \rightarrow M_{c'}$ is a Nash diffeomorphism. We will prove that $\log c' / \log c$ is rational. As g is one-to-one, for any $z \in S^{n-1}$ except at most one point, $\text{im } g \circ \phi_{cz}^n$ does not contain $0 \in M_{c'}$. It then follows from Remark 2.2 and the infinite vibration property of ϕ_{cz}^n that $p_{c'} \circ g \circ \phi_{cz}^n$ is constant (say z'). Clearly, the images of $q^{-1}(z' \times N_2)$ and $q^{-1}(z' \times (N_1 - (-\infty, -1]))$ in $M_{c'}$ are affine Nash submanifolds. Hence, $g \circ \rho_c^n \circ q^{-1}(z \times \overline{N_3}) = \text{img} \circ \phi_{cz}^n$ is not contained in the affine Nash submanifolds, which implies that $g \circ \rho_c^n \circ q^{-1}(z \times \overline{N_3})$ contains $\rho_{c'}^n \circ q^{-1}(z' \times \overline{N_3})$. Apply the same argument to g^{-1} . Then, $g^{-1} \circ \rho_{c'}^n \circ q^{-1}(z' \times \overline{N_3})$ contains $\rho_c^n \circ q^{-1}(z \times \overline{N_3})$. Therefore,

$$g \circ \rho_c^n \circ q^{-1}(z \times \overline{N_3}) = \rho_{c'}^n \circ q^{-1}(z' \times \overline{N_3}).$$

Then, in the same way as I, we see that $\log c' / \log c$ is rational.

III. The general compact case of M . By [5] any compact C^∞ manifold in $\mathbf{R}^m$ can be approximated by a Nash one in the C^∞ topology. Hence we assume that M is an affine Nash manifold containing 0 and contained in $\mathbf{R}^m$ and that $T_0M^\perp$ is transversal to M , where $T_0M^\perp$ is the space of normal vectors of M at 0 in $\mathbf{R}^m$. Set $A = T_0M^\perp \cap M$ (which is a finite set because A is a compact manifold of dimension 0 by the transversality). Let S^{n-1} , $n = \dim M$, be a sphere in the tangent space T_0M with center at 0, and let $r: \mathbf{R}^m \rightarrow T_0M$ and $p_1: T_0M - 0 \rightarrow S^{n-1}$ be the orthogonal projection and the natural projection, respectively. Set $p = p_1 \circ r$ on $M - A$. Then p is a Nash map, and there exists an open neighborhood U of 0 in M such that $r|_U: U \rightarrow r(U)$ is a Nash diffeomorphism. Let $E(\varepsilon)$ be the solid open sphere in T_0M of radius ε with center at 0. For simplicity we assume $\overline{E(2)} \subset r(U)$. Set

$$N_1'^n = r^{-1}E(1+d) \cap U, \quad N_2'^n = M - r^{-1}E(1) \cap U, \quad N_3'^n = r^{-1}E(2) \cap U - \overline{r^{-1}E(1)},$$

and in the same way as II define $h_1'^n: N_3'^n \rightarrow N_1'^n$, $h_2'^n: N_3'^n \rightarrow N_2'^n$, and an abstract Nash manifold M_c by patching $h_1'^n$ and $h_2'^n$. Then the proofs of (3.1) and (3.2) in II remain true in this case except for the statement in II stating that for any $z \in S^{n-1}$, except at most one point, $\text{im } g \circ \phi_{cz}^n$ does not contain 0. We have to replace this by saying that for any $z \in S^{n-1}$, except at most $\#A$ points, $\text{im } g \circ \phi_{cz}^n$ does not contain A . We omit the details.

IV. The general noncompact case of M . If M is not compact, we can consider it to be the interior of a compact affine Nash manifold with boundary in $\mathbf{R}^m$ [7]. Hence the argument in III holds true.

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