THE 3-DIMENSIONALITY OF CERTAIN CODIMENSION-3 DECOMPOSITIONS

R. J. DAVERMAN

ABSTRACT. It is proved that if p is a proper mapping of an $(n + 3)$ -manifold M onto a metric space B such that each inverse set $p^{-1}b$ has the shape of a closed, connected, orientable n -manifold, then B is 3-dimensional.

1. Introduction. If G is an upper semicontinuous decomposition of an $(n + k)$ -manifold M into continua having the shape of closed, orientable n -manifolds, what can be said about the decomposition space $B = M/G$? This question has been studied in several recent papers, most of which address low-dimensional cases, where $k < 3$. The present paper treats the case $k = 3$ and establishes, as its Main Result, that B is 3-dimensional.

Stronger conclusions have been obtained when $k < 3$. In case $k = 1$, the author has shown [D] that B is a 1-manifold (possibly with boundary if M is nonorientable). Similarly, in case $k = 2$, Daverman and Walsh [DW1] have proved that B is a 2-manifold (again, possibly with boundary if M is nonorientable). However, for $k > 2$, examples abound (see [DW2]) revealing that B need be neither a k -manifold nor, more generally, a generalized k -manifold. Its only known unilaterally held property, which happens to be crucial for what lies ahead, is the fact that B has cohomological dimension k [DW2].

In its conclusion the Main Result resembles the theorem of Kozłowski and Walsh [KW] that cell-like decompositions of 3-manifolds give rise to 3-dimensional spaces. Walsh's alternative proof [W2] of their result inspired the one given here. The argument also depends upon methods developed by Edwards [E] in his characterization of the cell-like images of manifolds that are manifolds, which prompted the device used in §3 for approximating the given decomposition map by one to which the arguments of Walsh would apply.

One can better appreciate the strong connection between the Main Result and results about the dimension-preserving aspects of cell-like decompositions by supposing there exists a cell-like decomposition G of some k -manifold N for which $\dim(N/G) > k$. Setting $M = N \times S^n$ and

$$G' = G \times S^n = \{g \times S^n | g \in G\},$$

one will obtain a decomposition of M into continua with the shape of S^n for which $\dim(M/G') > k$, because N/G and M/G' are naturally homeomorphic.

Received by the editors January 16, 1984.

1980 *Mathematics Subject Classification.* Primary 54B15, 55M10, 57N15; Secondary 54C56, 54F45.

Key words and phrases. Cohomological dimension, upper semicontinuous decomposition, proper map, codimension-3 manifold.

© 1986 American Mathematical Society
0002-9939/86 \$1.00 + \$.25 per page

The argument here is split into two segments, along the lines suggested in the tracing of antecedents. Identified in the next section is a geometric property of maps $q: M \rightarrow B$ implying the desired conclusion. Then, in the subsequent section, it is shown how to approximate the given decomposition map $p: M \rightarrow B$ by a map q satisfying the identified property.

2. Dimension and cohomological dimension. Throughout this section we assume the given $(n + 3)$ -manifold M to be orientable.

First, we state a fundamental technical result, taken from [DW2, Proposition 2.2]. It immediately implies the cohomological dimension of B is 3; whether this means B actually has dimension 3 is unknown (cf. [W3, p. 109]).

PROPOSITION 1. *Suppose q is a proper map of M to a metric space B such that, for each $b \in B$, $q^{-1}b$ has the shape of a closed, connected, orientable n -manifold, and suppose F is a closed subset of B such that $\dim q^{-1}F \leq n + m$. Then F has cohomological dimension at most m .*

Applications of Proposition 1 will be confined to the case $m = 1$. Then, based on the widely known relationship between dimension 1 and cohomological dimension 1 (cf. [W3, Corollary 3.3]), we have the following:

COROLLARY 2. *Under the hypotheses of Proposition 1, if F satisfies $\dim q^{-1}F \leq n + 1$, then $\dim F \leq 1$.*

In the space of all maps $[0, 1] \rightarrow M$ (endowed with the sup-norm metric), choose a countable dense subset consisting of pairwise disjoint, locally flat embeddings. Let $A_1, A_2, \dots, A_k, \dots$ denote their images.

The proof of the next result is an adaptation of one given by Walsh [W2].

PROPOSITION 3. *Suppose $q: M \rightarrow B$ is a proper surjective mapping satisfying the following:*

- (1) *each $q^{-1}b$ has the shape of a closed, connected, orientable n -manifold, and*
- (2) *$\dim q(A_k) \leq 1$ for $k = 1, 2, \dots$*

Then $\dim B \leq 3$.

PROOF. The 1-dimensional subset $\bigcup q(A_k)$ is contained in a 1-dimensional G_δ -subset $\bigcap W_j$, where each W_j is open in B (cf. [HW, p. 30]). No $q^{-1}(B - W_j)$ separates any connected open subset of M , essentially because it misses every A_k . As a result, $\dim q^{-1}(B - W_j) \leq n + 1$ [B, p. 14].

By Corollary 2, $\dim(B - W_j) \leq 1$. Expressing B as $(\bigcap W_j) \cup [\bigcup (B - W_j)]$, we can apply the Sum Theorem of [HW, p. 28] to conclude that $\dim B \leq 3$.

3. Improvements to the decomposition map.

PROPOSITION 4. *Let G be an upper semicontinuous decomposition of the orientable $(n + k)$ -manifold M into continua having the shape of closed, orientable n -manifolds. Then the decomposition map $p: M \rightarrow B = M/G$ can be approximated, arbitrarily closely, by a map $q: M \rightarrow B$ satisfying the following:*

- (1) *each $q^{-1}b$ has the shape of a closed, connected, orientable n -manifold, and*
- (2) *$\dim q(A_k) \leq 1$, for each of the locally flat arcs A_k in M named in §2.*

PROOF. In the space of all maps $M \rightarrow B$, let Q denote the closure of $\{ph|h: M \rightarrow M \text{ is a homeomorphism}\}$. If M is compact, the case on which one should focus, the sup-norm metric causes Q to be complete metric; in any event the limitation topology described by Toruńczyk [T] turns Q into a Baire space, which is all that matters. Consider

$$R_{m,i} = \{ q \in Q \mid \text{there exists a } (1/m)\text{-map of } q(A_m) \text{ to a 1-complex} \}.$$

Each $R_{m,i}$ clearly is an open subset of Q . Our primary goal is to show it is dense there, for by the Baire property it then will follow that $p \in Q$ can be approximated by $q \in \bigcap_m \bigcap_i R_{m,i}$, which necessarily must satisfy conclusion (2).

The rest of the argument is aimed at proving $R_{m,i}$ is dense in Q . Towards that end, fix a homeomorphism $\psi: [0, 1] \rightarrow A_i$.

Near each $q \in Q$ is some ph' : $M \rightarrow B$ (h' : $M \rightarrow M$ a homeomorphism). We shall show that near ph' is some $ph'h$ (h : $M \rightarrow M$ another homeomorphism) such that $ph'h(A_i)$ admits a $(1/m)$ -map to a 1-complex. For notational simplicity we suppress h' and study p itself rather than the equivalent ph' . Focusing on a compact neighborhood C of $p(A_i)$, we obtain h with support in $p^{-1}C$; to control the approximation it thus suffices to consider $\varepsilon > 0$ and to find h satisfying $\rho(p, ph) < \varepsilon$.

For $t \in [0, 1]$ let N_t denote the $\varepsilon/2$ -neighborhood of $p\psi(t)$, and let W_t denote another neighborhood of $p\psi(t)$ in N_t such that $p^{-1}W_t$ shape deformation retracts to $p^{-1}(p\psi(t))$ in $p^{-1}N_t$. For each $t \in [0, 1]$ choose a connected neighborhood U_t of $\psi(t)$ in A_i with $p(U_t) \subset W_t$.

Extract a finite open cover $\{U_{t(j)} \mid j = 0, 1, \dots, k\}$ of A_i from $\{U_t\}$. Without loss of generality assume no proper subcollection of $\{U_{t(j)}\}$ covers A_i and relabel so $t(j-1) < t(j)$ for $0 < j \leq k$. Set $s(0) = 0$ and $s(k+1) = 1$. Find points $s(j) \in (t(j-1), t(j))$ for $j \in \{1, \dots, k\}$ satisfying

$$(*) \quad \psi([s(j), s(j+1)]) \subset p^{-1}(W_{t(j)}) \quad \text{for } j \in \{0, 1, \dots, k\}.$$

We can obtain an arc B_j in $p([s(j), t(j)])$, since it is arcwise connected, from $p(s(j))$ to $p(t(j))$; similarly, we can obtain an arc B'_j in $p([t(j), s(j+1)])$ from $p(t(j))$ to $p(s(j+1))$. The resulting 1-dimensional set $X = (\bigcup B_j) \cup (\bigcup B'_j)$ admits a $(1/m)$ -map f to some 1-complex K , and f extends to a $(1/m)$ -map $f: Y \rightarrow K$, where Y represents a neighborhood of X in C .

By (*) and the choices of B_j and B'_j , we have $p^{-1}(B_j \cup B'_j) \subset p^{-1}(W_{t(j)})$ for $j \in \{0, 1, \dots, k\}$. Near each of the connected sets $p^{-1}(B_j \cup B'_j)$ is a path in $p^{-1}(W_{t(j)} \cap Y)$ from $\psi(s(j))$ to $\psi(s(j+1))$ and containing $\psi(t(j))$. Piecing these together, we produce a map $\psi': [0, 1] \rightarrow p^{-1}(Y)$ such that $\psi'(s(j)) = \psi(s(j))$ for $j \in \{0, 1, \dots, k+1\}$, whereas for $j \leq k$, $\psi'(t(j)) = \psi(t(j))$ and

$$\psi'([s(j), s(j+1)]) \subset p^{-1}(W_{t(j)}).$$

The major problem still to be faced is that ψ' might fail to be homotopic to ψ . For $j \in \{0, 1, \dots, k\}$ the maps $\psi'|[s(j), s(j+1)]$ and $\psi|[s(j), s(j+1)]$ can be combined, since they agree at their endpoints, to define a loop L_j in $p^{-1}(W_{t(j)})$. Although L_j need not be contractible in M , it does shape deform in $p^{-1}(N_{t(j)})$ to a loop L'_j in $p^{-1}(p\psi(t(j)))$. Consequently, we can redefine ψ' near $t(j)$ to trace out the inverse of L'_j , and then the new loop L_j , obtained from the modified ψ' , will be contractible in $p^{-1}(N_{t(j)})$.

With a final adjustment, we make $\psi: [0, 1] \rightarrow p^{-1}(Y)$ a locally flat embedding, homotopic rel $\{0, 1\}$ to ψ via a controlled homotopy θ_t with $\rho(p\theta_t, p\psi) < \varepsilon$ and $p\theta_t(s) \in C$ for all $s, t \in [0, 1]$. The Bing-Kister isotopy construction technique [BK, Theorem 5.5] provides an ambient isotopy approximately covering θ_t , whose end is a homeomorphism $h: M \rightarrow M$ such that $h\psi = \psi'$, the support of h is contained in $p^{-1}(C)$, and $\rho(p, ph) < \varepsilon$. This completes the proof that $R_{m,i}$ is dense in Q .

Shifting points of view, we now point out that, based on the above, we have shown the resulting map q satisfying conclusion (2) can be obtained as $\lim_{t \rightarrow \infty} ph_t$, where $h_t: M \rightarrow M$ is an isotopy defined for all $t \geq 0$, with h_0 the identity. An argument outlined by Walsh [W1, p. 124] (in case M is compact; the noncompact case admits similar controls) certifies that each $q^{-1}b$ has the same shape as $p^{-1}b$. As a result, q satisfies both (1) and (2). $\square$

MAIN RESULT. *If G is an upper semicontinuous decomposition of an $(n + 3)$ -manifold M into continua having the shape of closed, orientable n -manifolds, then the decomposition space M/G is 3-dimensional.*

PROOF. When M is orientable, this is a direct consequence of Propositions 3 and 4.

When M is nonorientable, we consider the orientable double covering $F: M' \rightarrow M$ and the partition G' of M' into the components of the various $F^{-1}(g)$, $g \in G$. Since G' is an upper semicontinuous decomposition of the type desired, $\dim(M'/G') \leq 3$. Moreover, the covering homeomorphism $h': M' \rightarrow M'$ preserves elements of G' , so h' induces an involution $u: M'/G' \rightarrow M'/G'$. The orbit space of u is equivalent to M/G , and the fixed point set X of u corresponds to the elements of G' on which h' is setwise invariant. Consequently, we have a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{F} & M \\ P' \downarrow & & P \downarrow \\ M'/G' & \xrightarrow{T_u} & M/G \end{array}$$

where $T_u: M'/G' \rightarrow M/G$ is essentially the map to the orbit space of u . Then $\dim T_u(X) \leq 3$, because $T_u|_X$ is 1-1, and $\dim T_u(M'/G' - X) \leq 3$, because $u|(M'/G' - X)$ is a covering map. The observation that $T_u(X)$ is closed clinches the result that $\dim M/G \leq 3$ [HW, p. 32].

REFERENCES

- [BK] R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes*, Duke Math. J. **31** (1964), 491–511.
- [B] A. Borel, *Seminar on transformation groups*, Ann. of Math. Stud., No. 46, Princeton Univ. Press, Princeton, N. J., 1960.
- [D] R. J. Daverman, *Decompositions of manifolds into codimension one submanifolds*, Compositio Math. (to appear).
- [DW1] R. J. Daverman and J. J. Walsh, *Decompositions of manifolds into codimension two submanifolds*, Trans. Amer. Math. Soc. **288** (1985), 273–291.
- [DW2] ———, *Decompositions into submanifolds that yield generalized manifolds* (in preparation).

- [E] R. D. Edwards, *The topology of manifolds and cell-like maps*, Proc. Internat. Congr. Math. Helsinki, 1978 (O. Lehto, ed.), Acad. Sci. Fenn. Helsinki, 1980, pp. 111–127.
- [HW] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.
- [KW] G. Kozłowski and J. J. Walsh, *Cell-like mappings on 3-manifolds*, *Topology* **22** (1983), 147–151.
- [T] H. Toruńczyk, *On CE images of the Hilbert cube and characterizations of Q-manifolds*, *Fund. Math.* **106** (1980), 31–40.
- [W1] J. J. Walsh, *Isotoping mappings to open mappings*, *Trans. Amer. Math. Soc.* **250** (1979), 121–145.
- [W2] ———, *The finite dimensionality of integral homology 3-manifolds*, *Proc. Amer. Math. Soc.* **88** (1983), 154–156.
- [W3] ———, *Dimension, cohomological dimension, and cell-like mappings*, *Shape Theory and Geometric Topology* (S. Mardesic and J. Segal, Eds.), *Lecture Notes in Math.*, vol. 870, Springer-Verlag, Berlin, 1981, pp. 105–118.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996-1300