SHORTER NOTES

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A UNIQUENESS THEOREM FOR MONIC BLASCHKE PRODUCTS

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ABSTRACT. If two monic Blaschke products of order n agree at n points of the open unit disc $\mathbf{D}$, then they must be identical.

THEOREM. *Let*

$$A(z) = \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \quad \text{and} \quad B(z) = \prod_{j=1}^n \frac{z - b_j}{1 - \bar{b}_j z},$$

with a_j and $b_j \in \mathbf{D} = \{ |z| < 1 \}$ for $j = 1, \dots, n$. Suppose that $A(\lambda_j) = B(\lambda_j)$ for n distinct points $\lambda_1, \dots, \lambda_n$ in $\mathbf{D}$. Then $A \equiv B$.

REMARK. The restriction that A and B be monic (i.e. that the normalizing constants be 1) is essential, as the following simple example shows. Take $|c| = 1$, $c \neq 1$ and let

$$A(z) = c \prod_{j=1}^n \frac{z - a_j}{1 - \bar{a}_j z} \quad \text{and} \quad B(z) = \frac{z - ca_1}{1 - \bar{c}\bar{a}_1 z} \prod_{j=2}^n \frac{z - a_j}{1 - \bar{a}_j z},$$

supposing now that $a_j \neq 0$ for all j . Then $A(a_j) = B(a_j)$ for $j = 2, \dots, n$ and $A(0) = B(0)$ but $A \not\equiv B$. Indeed, $A \not\equiv kB$ for any constant k .

LEMMA. *Let $A(z)$ and $B(z)$ be as in the statement of the Theorem. Then $A(\lambda) = B(\lambda)$ for some λ with $|\lambda| = 1$.*

PROOF. For $A(z) = B(z)$ iff

$$(1) \quad \prod_{j=1}^n \left(\frac{z - a_j}{z - b_j} / \frac{1 - \bar{a}_j z}{1 - \bar{b}_j z} \right) = 1$$

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and if $|z| = 1$, this says

$$(2) \quad \prod_{j=1}^n \left[\frac{z - a_j}{z - b_j} / \overline{\left(\frac{z - a_j}{z - b_j} \right)} \right].$$

Now $w/\bar{w} = \exp 2i \arg w$, so we will be done if we can prove that, for some determination of the argument, for some z with $|z| = 1$, and for some integer m ,

$$(3) \quad \arg \prod_{j=1}^n \frac{z - a_j}{z - b_j} = \pi m.$$

But this is easy. Let

$$(4) \quad F(z) = \prod_{j=1}^n \frac{z - a_j}{z - b_j}.$$

Then $F(z)$ is analytic and zero-free for $|z| > 1 - \delta$ for some positive δ . Hence there is (see [1, p. 204]) an analytic branch of $H(z) = \log F(z)$ for $|z| > 1 - \delta$ (which includes the point at infinity.) Furthermore, $F(\infty) = 1$, so we may choose $H(\infty) = 0$. By the Gauss mean-value property of harmonic functions (after the change of variables $z \rightarrow 1/z$, perhaps), we see that $\operatorname{Im} H(\lambda) = 0$ for some λ on $|\lambda| = 1$. But it is easy to see that the harmonic function $\operatorname{Im} H(z)$ is a branch of $\arg F(z)$. Since any two continuous branches of $\arg F(z)$ differ by a constant integer multiple of 2π , the lemma is proved.

REMARK. The assertion (3) has an appealing *geometric interpretation*, so it seems likely that there is a purely geometric proof of it. Place any finite number of directed line segments $L_j = \overline{a_j b_j}$ in the unit disc $\mathbf{D}$, $j = 1, \dots, n$. There is no restriction on their number, their length, or their placement. Let $\varphi = \varphi(\theta)$ be the (directed) angle the union of these segments subtends from the point $e^{i\theta}$ on the unit circle. That is,

$$\varphi(\theta) = \sum_{j=1}^n \left[\arg(e^{i\theta} - b_j) - \arg(e^{i\theta} - a_j) \right]$$

for some continuous determination of the argument, as θ runs from 0 to 2π . Then there must be at least one value of θ in this range for which $\varphi(\theta)$ is an integral multiple of π .

PROOF OF THEOREM. Let $R(z) = A(z)/B(z)$. Since $|R(z)| = 1$ for all $|z| = 1$, we have

$$R(z) \overline{R}(1/\bar{z}) = 1$$

first on $\{|z| = 1\}$ and then for all z on the Riemann sphere, by analytic continuation. Since $R(\lambda_j) = 1$, we have $R(1/\bar{\lambda}_j) = 1$ for $j = 1, \dots, n$. By the Lemma, $R(\lambda) = 1$ for some λ on $|\lambda| = 1$. Thus the rational function $R(z)$, of degree $2n$, takes on the value 1 at the $2n + 1$ distinct points $\lambda, \lambda_1, \dots, \lambda_n, 1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ on the Riemann sphere. (Since each $|\lambda_j| < 1$, each $|1/\bar{\lambda}_j| > 1$.) Hence $R(z) \equiv 1$, and $A(z) \equiv B(z)$ as desired.

It seems natural in the context of our Theorem to guess that a value λ_j with $|\lambda_j| = 1$ is “worth half” a λ_j with $|\lambda_j| < 1$, but this just is not so. Choose a and b

with $0 < a < b < 1$, and let $A(z) = (z - a)/(1 - az)$, $B(z) = (z - b)/(1 - bz)$. Then $A(1) = B(1)$ and $A(-1) = B(-1)$ yet surely *not* $A \equiv B$. This is the case $n = 1$. For general n , we let w_j run through the n th roots of 1 and v_j run through the n th roots of -1 , and then let $a_j = aw_j$ and $b_j = bw_j$, and define $A(z)$ and $B(z)$ as earlier. Then $A(w_j) = B(w_j)$ for each $j = 1, 2, \dots, n$, and $A(v_j) = B(v_j)$ for each $j = 1, 2, \dots, n$, making $2n$ distinct points of coincidence of A and B on $\partial\mathbf{D}$. Yet $A \not\equiv B$. The details of the computations are easy, and are left to the reader.

The above "paradox" disappears on further considering our proof of the Theorem. If all the λ_j are taken to lie inside $\mathbf{D}$, then Lemma 1 produces an *extra* point λ of coincidence on the boundary. But if some λ_j lies on the boundary of $\mathbf{D}$, then we might well have $\lambda = \lambda_j$, so that no extra point is produced.

REFERENCES

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