COEFFICIENTS OF ODD UNIVALENT FUNCTIONS

HU KE

ABSTRACT. Let

$$S_2 = \left\{ f_2(z) = z + \sum_{n=1}^{\infty} b_n z^{2n+1} \in S \right\}.$$

In this note we prove $|b_n| < 1.1305$. This is an improvement of V. I. Milin's result [1].

Let

$$S_2 = \left\{ f_2(z) = z + \sum_{n=1}^{\infty} b_n z^{2n+1} \mid f_2 \text{ is analytic} \right. \\ \left. \text{and univalent in the unit disk } |z| < 1 \right\}.$$

The argument of Littlewood and Paley [4] gives $|b_n| < 14$, $n = 1, 2, \dots$. The bound cannot be reduced to 1 because there is a function for which $|b_5| = 1/2 + e^{-2/3} > 1$. It was improved by Levin [3] to 3.39 and by Kung Sun [5] to 2.54. The work of I. M. Milin [2] improved it to 1.17, and V. I. Milin [1] has further improved it to 1.14. In this note we give a slight improvement of the above result.

THEOREM. *If $f_2(z) = z + \sum_{n=1}^{\infty} b_n z^{2n+1}$, then $|b_n| < 1.1305$.*

PROOF. Our proof depends on Milin's Conjecture. Let $f_2 \in S$, and

$$\log \left\{ \frac{f_2(\sqrt{z})}{\sqrt{z}} \right\} = \sum_{n=1}^{\infty} c_n z^n.$$

Then

$$(1) \quad \sum_{k=1}^n (n+1-k)k|c_k|^2 \leq \sum_{k=1}^n \frac{n+1-k}{k}, \quad n = 2, 3, \dots,$$

which has been proved by de Branges in [6].

Received by the editors December 17, 1984 and, in revised form, March 6, 1985. The contents of this paper have been presented to a scientific meeting of Jiangxi Teacher's University in October 15, 1984. 1980 *Mathematics Subject Classification*. Primary 30C50; Secondary 30C55.

Thus, for $0 < \rho < 1$, we have

$$\begin{aligned}
 (2) \quad \sum_{k=1}^n k|c_k|^2 \rho^k &= (1-\rho)^2 \left(\sum_{k=1}^n k|c_k|^2 \rho^k \right) \left(\sum_{k=0}^{\infty} (k+1) \rho^k \right) \\
 &= (1-\rho)^2 \left\{ \sum_{k=1}^n \left[\sum_{i=1}^k (k-i+1) i |c_i|^2 \right] \rho^k \right. \\
 &\quad \left. + \rho^n \sum_{\nu=1}^{\infty} \left(\sum_{i=1}^n (n+\nu-i+1) i |c_i|^2 \right) \rho^{\nu} \right\} \\
 &\leq (1-\rho^2) \left\{ \sum_{k=1}^n \left(\sum_{i=1}^k \frac{k-i+1}{i} \right) \rho^k + \frac{\rho^{n+1}}{1-\rho} \sum_{i=1}^n \frac{n-i+1}{i} \right. \\
 &\quad \left. + \frac{\rho^{n+1}}{(1-\rho)^2} \sum_{i=1}^n i |c_i|^2 \right\} \\
 &= \sum_{k=1}^n \frac{1}{k} \rho^k + \rho^{n+1} \sum_{i=1}^n \left(i |c_i|^2 - \frac{1}{i} \right) = I_{\rho} + \rho^{n+1} \delta,
 \end{aligned}$$

where δ (< 0.312) is the Milin constant [2].

We may apply the Lebedev-Milin inequality to conclude that

$$(3) \quad |b_n| \leq \exp \frac{1}{2} \left\{ \sum_{k=1}^n k|c_k|^2 \rho^k - \frac{1}{k} - n \log \rho \right\}.$$

Notice that

$$\frac{1}{e^x - e^{-x}} \geq \frac{1}{2x} e^{-x^2/6} > \frac{1}{2x} \left(1 - \frac{x^2}{6} \right)$$

and let $y = 0.4574$, $\rho = e^{-2y/(2n+1)}$. Then

$$\begin{aligned}
 (4) \quad \sum_{k=n+1}^{\infty} \frac{1}{k} \rho^k &= \frac{2}{2n+1} \int_y^{\infty} \frac{e^{-\tau}}{e^{\tau/(2n+1)} - e^{-\tau/(2n+1)}} d\tau \\
 &> \int_y^{\infty} \frac{e^{-\tau}}{\tau} \left(1 - \frac{\tau^2}{6(2n+1)^2} \right) d\tau \\
 &= \int_y^{\infty} \frac{e^{-\tau}}{\tau} d\tau - \frac{e^{-y}(1+y)}{6(2n+1)^2}.
 \end{aligned}$$

Since for $x > 0$,

$$\frac{1}{e^x - e^{-x}} \leq \frac{1}{2x(1 + \frac{1}{3}x^2)},$$

and for $t > 0$,

$$\log \frac{1}{1+t} = \left\{ \frac{t}{1+t} + \frac{1}{2} \left(\frac{t}{t+1} \right)^2 + \frac{1}{3} \left(\frac{t}{t+1} \right)^3 + \cdots \right\},$$

we get

$$\begin{aligned}
 (5) \quad \log \frac{1}{1-\rho} &= \log \frac{e^{y/(2n+1)}}{e^{y/(2n+1)} - e^{-y/(2n+1)}} \\
 &< \frac{y}{2n+1} + \log \left\{ \frac{2y}{2n+1} \left[1 + \frac{1}{3} \left(\frac{y}{2n+1} \right)^2 \right] \right\}^{-1} \\
 &\leq \frac{y}{2n+1} + \log \frac{1}{y} + \log \left(n + \frac{1}{2} \right) - \frac{y^2}{3(2n+1)^2 + y^2} \\
 &\leq \sum_{k=1}^n \frac{1}{k} - c + \log \frac{1}{y} + \frac{y}{2n+1} + \frac{y^2}{3(2n+1)^2 + y^2},
 \end{aligned}$$

where the inequality $\log(n + \frac{1}{2}) < \sum_{k=1}^n 1/k - c$ has been used, and c is the Euler constant.

Inequalities (4) and (5) lead to the estimate

$$\begin{aligned}
 (6) \quad \sum_{k=1}^n \frac{\rho^k}{k} &= \log \frac{1}{1-\rho} - \sum_{k=n+1}^{\infty} \frac{1}{k} \rho^k \\
 &\leq \sum_{k=1}^n \frac{1}{k} - c - \int_y^{\infty} \frac{e^{-\tau}}{\tau} d\tau + \log \frac{1}{y} \\
 &\quad + \frac{y}{2n+1} + \frac{e^{-y}(1+y)}{6(2n+1)^2} - \frac{y^2}{3(2n+1)^2 + y^2}
 \end{aligned}$$

for $\rho = e^{-2y/(2n+1)}$ and $y = 0.4574$.

It is easy to deduce that, for $\rho = e^{-2y/(2n+1)}$ and $y = 0.4574$,

$$\begin{aligned}
 (7) \quad \sum_{k=1}^n k |c_k|^2 \rho^k - n \log \rho - \sum_{k=1}^n \frac{1}{k} \\
 &\leq \sum_{k=1}^n \frac{\rho^k}{k} + \delta \rho^{n+1} - n \log \rho - \sum_{k=1}^n \frac{1}{k} \\
 &\leq y + \log \frac{1}{y} - c - \int_y^{\infty} \frac{e^{-\tau}}{\tau} d\tau + \frac{e^y(1+y)}{6(2n+1)^2} - \frac{y^2}{3(2n+1)^2 + y^2} \\
 &\quad + 0.312 e^{-y-y/(2n+1)} \\
 &\leq y + \log \frac{1}{y} - c - \int_y^{\infty} \frac{e^{-\tau}}{\tau} d\tau + 0.312 e^{-y} \quad (n \geq 3) \\
 &= 0.4574 - 0.577215 + 0.167745 + 0.197473 = 0.245433.
 \end{aligned}$$

Substitute this result into (3) to deduce

$$|b_n| \leq e^{0.1227} < 1.1305, \quad n \geq 3.$$

It is known that $|b_1| \leq 1$, $|b_2| \leq 1.013$. Thus the proof of the theorem is complete.

REFERENCES

1. V. I. Milin, *Estimate of the coefficients of odd univalent functions*, Metric Questions of the Theory of Functions (G. D. Suvorov, ed.), "Naukova Dumka", Kiev, 1980, pp. 78–86. (Russian)
2. I. M. Milin, *Univalent functions and orthonormal systems*, "Nauka", Moscow, 1971; English transl. in Transl. Math. Monographs, vol. 49, Amer. Math. Soc., Providence, R. I., 1977.
3. V. I. Levin, *Some remarks on the coefficients of schlicht functions*, Proc. London Math. Soc. **39** (1935), 467–480.
4. J. E. Littlewood and R. E. A. C. Paley, *A proof that an odd schlicht function has bounded coefficients*, J. London Math. Soc. **7** (1932), 167–169.
5. Kung Sun, *Contributions to the theory of schlicht functions. II: The coefficient problem*, Sci. Sinica **4** (1955), 359–373.
6. L. de Branges, *A proof of the Bieberbach conjecture*, Steklov Mat. Inst., LOMI, preprint E-5-84, Leningrad, 1984, pp. 1–21.

DEPARTMENT OF MATHEMATICS, JIANGXI TEACHER'S UNIVERSITY, NANCHANG, CHINA