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NORMAL SUBGROUPS OF THE GENERAL LINEAR GROUPS OVER VON NEUMANN REGULAR RINGS

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ABSTRACT. Let A be a von Neumann regular ring or, more generally, let A be an associative ring with 1 whose reduction modulo its Jacobson radical is von Neumann regular. We obtain a complete description of all subgroups of $\mathrm{GL}_n A$, $n \geq 3$, which are normalized by elementary matrices.

1. Introduction. For any associative ring A with 1 and any natural number n, let $\mathrm{GL}_n A$ be the group of invertible n by n matrices over A and $E_n A$ the subgroup generated by all elementary matrices $x^{i,j}$, where $1 \leq i \neq j \leq n$ and $x \in A$.

In this paper we describe all subgroups of $\mathrm{GL}_n A$ normalized by $E_n A$ for any von Neumann regular A, provided $n \geq 3$. Our description is standard (see Bass [**1**] and Vaserstein [**14, 16**]): a subgroup H of $\mathrm{GL}_n A$ is normalized by $E_n A$ if and only if H is of *level* B for an ideal B of A, i.e. $E_n(A, B) \subset H \subset G_n(A, B)$. Here $G_n(A, B)$ is the inverse image of the center of $\mathrm{GL}_n(A/B)$ (when $n \geq 2$, this center consists of scalar invertible matrices over the center of the ring A/B) under the canonical homomorphism $\mathrm{GL}_n A \to \mathrm{GL}_n(A/B)$ and $E_n(A, B)$ is the normal subgroup of $E_n A$ generated by all elementary matrices in $G_n(A, B)$ (when $n \geq 3$, the group $E_n(A, B)$ is generated by matrices of the form $(-y)^{j,i} x^{i,j} y^{j,i}$ with $x \in B, y \in A, 1 \leq i \neq j \leq n$, see [**14**]).

Recall that a ring A is called von Neumann regular (see von Neumann [**13**], Goodearl [**7**]) if for any z in A there is x in A such that $zxz = z$. Then every factor ring and every ideal of A is also von Neumann regular.

In fact, to be more general, we assume that $A/\mathrm{rad}(A)$ (rather than A) is von Neumann regular, where rad means the Jacobson radical. For example, this assumption holds for any Artinian ring A or for any commutative semilocal ring A.

THEOREM 1. *Assume that $A/\mathrm{rad}(A)$ is von Neumann regular and $n \geq 2$. Then for any ideal B of A:*

(a) *$E_n(A, B)$ contains all matrices of the form $1_n + vu$, where v is an n-column over A, u is an n-row over B, and $uv = 0$; in particular, $E_n(A, B)$ is normal in $\mathrm{GL}_n A$;*

(b) *$E_n(A, B) \supset [E_n A, G_n(A, B)]$; in particular, every subgroup of $\mathrm{GL}_n A$ of level B is normalized by $E_n A$;*

(c) *if $n \geq 3$, we have $E_n(A, B) = [E_n A, E_n B] = [\mathrm{GL}_n A, E_n(A, B)] = [E_n A, H]$ for any subgroup H of level B, where $E_n B$ is the subgroup of $G_n(A, B)$ generated by elementary matrices;*

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(d) *if A is von Neumann regular, we have $E_n B = E_n(A, B)$; if moreover, $n \geq 3$, we have $E_n B = [E_n B, E_n B]$.*

THEOREM 2. *Assume that $A/\mathrm{rad}(A)$ is von Neumann regular and $n \geq 3$. Then every subgroup H of $\mathrm{GL}_n A$ normalized by $E_n A$ is of level B for some ideal B of A, i.e. $E_n(A, B) \subset H \subset G_n(A, B)$.*

Note that a subgroup H of $\mathrm{GL}_n A$, $n \geq 2$, cannot be of level B and of level B' for two distinct ideals B and B' of A. So the level B in Theorem 2 is unqiue.

Theorems 1 and 2 were proved by Dickson [**2**] when A is a field (the condition $n \geq 3$ in this case can be replaced by the condition $\mathrm{card}(A) \geq 4$), by Dieudonné [**3**] when A is a division ring, by Klingenberg [**10**] when A is a commutative local ring, by Bass [**1**] when A satisfies the stable range condition $\mathrm{sr}(A) \leq n-1$, by Vaserstein [**14**] when central localizations of A satisfy this stable range condition (for example, when A is finite as module over its center) and $n \geq 3$, and by Vaserstein [**16**] when A is a Banach algebra. Theorem 2 is claimed by Golubchik [**5**, **6**] under the additional condition that A/M is an Ore ring for every maximal ideal M of A.

Note that von Neumann regular rings A satisfying $\mathrm{sr}(A) \leq 1$ are known as unit regular rings, see [**7**, **8**, **9**, **11**, **12**, **15**].

2. Proof of Theorem 1(a). We write

$$v = (v_i) = \begin{pmatrix} v' \\ v_n \end{pmatrix} \quad \text{and} \quad u = (u_j) = (u', u_n)$$

with v_i in A and u_j in B.

Case 1. $1 + v_n u_n \in \mathrm{GL}_1 B$. We set $d := 1 + v_n u_n$, $d' := 1 + u_n v_n = 1 - u'v' \in \mathrm{GL}_1 B$ (see [**17**, §2]) and $a = 1_{n-1} + v'u' - v'u_n d^{-1} v_n u' = 1_{n-1} + v'(1 - u_n d^{-1} v_n) u' = 1_{n-1} + v'd'^{-1}u'$. Then

$$\begin{aligned} 1_n + vu &= \begin{pmatrix} 1_{n-1} + v'u' & v'u_n \\ v_n y' & d \end{pmatrix} \\ &= \begin{pmatrix} 1_{n-1} & v'u_n d - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ d^{-1} v_n u' & 1 \end{pmatrix} \\ &\in E_n B \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} E_n B. \end{aligned}$$

We have to prove that $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in E_n(A, B)$.

Since $1 + u'v'd'^{-1} = d'^{-1}$, we have

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1_{n-1} & 0 \\ u'a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & -v'd'^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ -u' & 1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1_{n-1} & v' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ 0 & d'^{-1} \end{pmatrix} \\ &\in E_n(A, B) \begin{pmatrix} 1_{n-1} & 0 \\ 0 & d'^{-1} \end{pmatrix}. \end{aligned}$$

By [**17**, §2],

$$\begin{pmatrix} 1_{n-1} & 0 \\ 0 & d'^{-1}d \end{pmatrix} \in E_n(A, B).$$

So $1_n + vu \in E_n(A, B)$ in Case 1.

Case 2. $v_i \in \operatorname{rad}(A)$ for some i with $1 \le i \le n$. Since $E_n(A, B)$ is normalized by all permutation matrices, we can assume that $i = n$. Then $1 + v_n u_n \in \mathrm{GL}_1 B$, so we are reduced to Case 1.

General case. We now use the condition that $A/\operatorname{rad}(A)$ is von Neumann regular, hence there is an x in A such that $v_n x v_n - v_n \in \operatorname{rad}(A)$. Then $1 + v_n(1 - xv_n)u_n \in \mathrm{GL}_1 B$, hence $g := 1_n + v(1 - xv_n)u \in \mathrm{GL}_n B$ by Case 1.

Also we have

$$\begin{aligned}(-v_{n-1}x)^{n-1,n}&(1_n + vxv_n u)(v_{n-1}x)^{n-1,n} \\ &= 1_n + ((-v_{n-1}x)^{n-1,n} vxv_{n-1})(u(v_{n-1}x)^{n-1,n})\end{aligned}$$

and

$$((-v_{n-1}x)^{n-1,n} vxv_n)_{n-1} = v_{n-1}(1 - xv_n)xv_n = v_{n-1}x(v_n - v_n x v_n) \in \operatorname{rad}(A),$$

hence $h := 1_n + vxv_n u \in E_n(A, B)$ by Case 2 with $i = n - 1$.

Therefore $1_n + vu = gh \in E_n(A, B)$.

3. Proof of Theorem 1(b). It suffices to show that $[y^{i,j}, g] := y^{i,j} g(-y)^{i,j} g^{-1} \in E_n(A, B)$ for any elementary $y^{i,j}$ in $E_n A$ and any g in $G_n(A, B)$. Since $E_n(A, B)$ is norr alized by all permutation matrices, we can assume that $(i, j) = (1, n)$.

Then $[y^{i,j}, g] = y^{1,n}(1_n - vyw)$, where $v = \binom{v'}{v_n}$ is the first column of g and $w = (w', w_n)$ is the last row of g^{-1}, so $wv = 0$.

As in the end of the previous section, we find x in A such that $v_n x v_n - v_n \in \operatorname{rad}(A)$, and we have $h := 1_n - vxv_n w \in E_n(A, B)$, hence

$$[(xv_n)^{1,n}, g] = (xv_n)^{1,n}(1_n - vxv_n w) \in E_n(A, B),$$

i.e. $(xv_n)^{1,n}$ and g commute modulo $E_n(A, B)$.

To complete our proof, it suffices to show that $(1 - xv_n)^{1,n}$ also commutes with g modulo $E_n(A, B)$. We set $u := -(1 - xv_n)w = (u', u_n)$. Then

$$[(1 - xv_n)^{1,n}, g] = (1 - xv_n)^{1,n}(1_n + vu),$$

with $v_n u_n = v_n(1 - xv_n)w_n \in \operatorname{rad}(B)$, hence $d := 1 + v_n u_n \in \mathrm{GL}_1 B$. Also $v_i \in B$ for $i \ge 2$, $u_j \in B$ for $j \le n - 1$ and $v_1 u_n + 1 \in B$.

We set $d' := 1 + u_n v_n = 1 - u'v' \in \mathrm{GL}_1 B$ and $a := 1_{n-1} + v'u' - v'u_n d^{-1} v_n u' = 1_{n-1} + v'd'^{-1}u'$. Then

$$\begin{aligned}(1 - xv_n)^{1,n}(1_n + vu) &= (1 - xv_n)^{1,n} \begin{pmatrix} 1_{n-1} & v'u_n d^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1_{n-1} & 0 \\ d^{-1}v_n u' & 1 \end{pmatrix} \\ &\in E_n B \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} E_n B.\end{aligned}$$

Now, as in the previous section (see Case 1 there), we see that $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in E_n(A, B)$. (Note that u' is an $(n-1)$-row over B.)

4. Proof of Theorem 1(c). In the view of Theorem 1(a), (b), we have only the inclusion $E_nB \subset [E_nA, E_nB]$ to prove. But we have it for any ring A with 1 and any $n \geq 3$ by the formula $x^{i,j} = [1^{i,k}, x^{k,j}]$, where $1 \leq i \neq j \neq k \neq i \leq n$ and $x \in B$.

5. Proof of Theorem 1(d). We want to prove first that $E_n(A, B) = E_nB$, i.e. E_nB is normalized by every elementary matrix $y^{i,j}$ in $\mathrm{GL}_n A$. Since E_nB is normalized by all permutation matrices, we can assume that $(i,j) = (1,2)$. It suffices to prove that $h := (-y)^{1,2}gy^{1,2} \in E_nB$ for every elementary matrix g in E_nB. This is trivial (and true for an arbitrary ring A) unless $g = z^{2,1}$ where $z \in B$. In this case we can assume that $n = 2$.

Since A is von Neumann regular, $z = zxz$ for some x in A. We have

$$h = (-y)^{1,2}z^{2,1}y^{1,2} = (-xzy)^{1,2}(xzy - y)^{1,2}z^{2,1}(y - xzy)^{1,2}(xzy)^{1,2}.$$

But $(xzy)^{1,2} \in E_2B$ and

$$\begin{aligned}(xzy - y)^{1,2}z^{2,1}(y - xzy)^{1,2} &= \begin{pmatrix} 1 + (xz - 1)yz & 0 \\ z & 1 \end{pmatrix} \\ &= ((xz - 1)yzx)^{1,2}z^{2,1}((1 - xz)yz)^{1,2} \in E_2B.\end{aligned}$$

When $n \geq 3$, for any elementary $z^{i,j}$ in E_nB we have $z^{i,j} = [(zx)^{i,k}, z^{k,j}]$, where $k \neq i, j$ and $z = zxz$ with x in A.

6. Proof of Theorem 2. Let H be a subgroup of $\mathrm{GL}_n A$ normalized by E_nA, where $n \geq 3$. The condition that $A/\mathrm{rad}(A)$ is von Neumann regular will not be used in Cases 1–5 of Lemma 3 below or Lemma 4.

LEMMA 3. *If H is not central, then H contains an elementary matrix $\neq 1_n$.*

PROOF. *Case* 1. $H \ni g = (g_{i,j})$ such that $g_{n,1} = 0$ and g does not commute with some $1^{k,1} \in E_nA$. Then H contains an elementary matrix $\neq 1_n$ by Vaserstein [**14**].

Case 2. $H \ni h = (h_{i,j})$ such that $h_{n,2} \neq 0$ and $h_{n,1} + h_{n,2}y = 0$ for some y in A. Then $H \ni (-y)^{2,1}hx^{2,1} =: g = (g_{i,j})$ and $g_{n,1} = h_{n,1} + h_{n,2}y = 0, g_{n,2} = h_{n,2} \neq 0$, so $[g, 1^{2,1}] \neq 1_n$. Thus, we are reduced to Case 1.

Case 3. H contains a noncentral $g = (g_{i,j})$ with $g_{n,1} = 0$. If g does not commute with some $1^{k,1} \in E_nA$, we are done by Case 1. Otherwise, g is a scalar matrix: $g_{i,j} = 0 = g_{i,i} - g_{j,j}$ for all $i \neq j$. Since g does not belong to the center of $\mathrm{GL}_n A$, there is y in A such that $yg_{1,1} \neq g_{1,1}y$. Then $[g, y^{1,2}] = (g_{1,1}y - yg_{2,2})^{1,2} \neq 1_n$ is an elementary matrix in H.

Case 4. H contains a noncentral $h = (h_{i,j})$ with $h_{2,2} \in \mathrm{GL}_1 A$. If $(h^{-1})_{n,1} = 0$, we are done by Case 3 with $g = h^{-1}$. Otherwise, $H \ni (-1)^{1,2}1^{1,2}h = (g_{i,j})$ with $(g_{n,1}, g_{n,2}) = (h^{-1})_{n,1}(h_{2,1}, h_{2,2})$, so we are reduced to Case 2.

Case 5. H contains a noncentral $h = (h_{i,j})$ with $h_{n,2} = 0$. Since $f := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in E_2A$, we have $f' := \left(\begin{smallmatrix} f & 0 \\ 0 & 1_{n-2} \end{smallmatrix}\right) \in E_nA$ and $g := f'hf'^{-1} \in H$. Since $g_{n,1} = h_{n,2} = 0$, we are reduced to Case 3.

General case. We pick a noncentral $h = (h_{i,j})$ in H and find x in A such that $z := h_{n,2}xh_{n,2} - h_{n,2} \in \mathrm{rad}(A)$. We set $p := 1 - h_{n,2}x$. If $ph_{n,1} = 0$, i.e. $h_{n,1} - h_{n,2}xh_{n,1} = 0$, then we are done by Case 5 or Case 2. Otherwise, the matrix

$g = (g_{i,j}) := h^{-1}p^{1,n}h(-p)^{1,n} \in H$ is not central and $g_{2,2} = 1 + (h^{-1})_{2,1}ph_{n,2} = 1 - (h^{-1})_{2,1}z \in \mathrm{GL}_1 A$, so we are reduced to Case 4.

LEMMA 4. *If $H \ni x^{i,j}$, where $x \in A, 1 \leq i \neq j \leq n$, then $H \supset E_n(A, B)$, where B is the (two-sided) ideal of A generated by x.*

PROOF. It follows easily from the identities $y^{i,j}z^{i,j} = (y+z)^{i,j}$ and $[y^{i,j}, z^{j,k}] = (yz)^{i,k}$, where $1 \leq i \leq j \neq k \neq i \leq n$ and y, z are in A (we use here that $n \geq 3$; no conditions on A are needed).

Now we can conclude our proof of Theorem 2. By Lemma 4, there is an ideal B of A such that $E_n(A, B)$ contains all elementary matrices in H. Consider the image H' of H in $\mathrm{GL}_n(A/B)$. Since the ring $(A/B)/\mathrm{rad}(A/B)$ is a factor ring of $A/\mathrm{rad}(A)$, it is also von Neumann regular. Since H' is normalized by $E_n(A/B)$ which is the image of E_nA, Lemma 3 applied to H' gives that either H' is central or H' contains an elementary matrix $(x')^{i,j}$, where $0 \neq x' \in A/B$ and $1 \leq i \neq j \leq n$. In the latter case, $H \ni x^{i,j}g$, where $0 \neq x \in A, x' = x + B$, and $g \in \mathrm{GL}_n B$. We pick an integer $k \neq i, j$ in the interval $1 \leq k \leq n$. Then $H \ni [x^{i,j}g, 1^{j,k}] = x^{i,k}1^{j,k}x^{i,j}[(-1)^{j,k}, g](-x)^{i,j}(-1)^{j,k} \in x^{i,k}E_n(A, B) \subset x^{i,k}H$ by Theorem 1(b). Therefore $H \ni x^{i,k}$ which contradicts our choice of B.

Thus, H' is central in $\mathrm{GL}_n(A/B)$, i.e. $H \subset G_n(A, B)$.

REMARK. From the proof of Theorem 1(a) (see §2 above), it is clear that the group $E_n(A, B)$ is generated by matrices of the form $(-y)^{j,i}x^{i,j}y^{j,i}$ with x in B and y in A, provided $n \geq 2$ and $A/\mathrm{rad}(A)$ is von Neumann regular. If $n \geq 3$, no restrictions on A are needed.

REFERENCES

1. H. Bass, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Etudes Sci. **22** (1964), 5–60.
2. L. E. Dickson, *Theory of linear groups in an arbitrary field*, Trans. Amer. Math. Soc. **2** (1901), 363–394.
3. J. Dieudonné, *Le géométrie des groupes classiques*, 3rd ed., Springer, New York and Berlin, 1971.
4. G. Ehrlich, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc. **216** (1976), 81–90.
5. I. Z. Golubchik, *Normal subgroups of linear groups over rings*, Vestnik Moskov Univ. Ser. I Mat. Mekh. **1978**, no. 6, 79. (Russian)
6. I. Z. Golubchik and A. V. Mikhalev, *Epimorphisms of projective groups over associative rings*, Algebra, Moskov Univ., 1982, pp. 34–45.
7. K. R. Goodearl, *Von Neumann regular rings*, Pitman, London, San Francisco and Melbourne, 1979.
8. M. Henriksen, *On a class of regular rings that are elementary divisor rings*, Arch. Math. **24** (1973), 133–141.
9. R. Hartwig and J. Luh, *A note on the group structure of unit regular ring elements*, Pacific J. Math. **71** (1977), 449–461.
10. W. Klingenberg, *Lineare Gruppen über lokalen Ringen*, Amer. J. Math. **83** (1961), 137–153.
11. P. Menal and J. Moncasi, *On regular rings with stable range* 2, J. Pure Appl. Algebra **24** (1982), 25–40.
12. ——, *K_1 of von Neumann regular rings*, J. Pure Appl. Algebra **33** (1984), 295–312.
13. J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 707–713.
14. L. N. Vaserstein, *On normal subgroups of* GL_n *over a ring*, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 456–465.
15. ——, *Bass's first stable range condition*, J. Pure Appl. Algebra **34** (1984), 319–330.

16. ——, *Normal subgroups of the general linear groups over Banach algebras*, preprint, Institute for Advanced Study, January 1985; J. Pure Appl. Algebra (to appear).
17. ——, *K_1-theory and the congruence subgroup problem*, Mat. Zametki **5** (1969), 233–244=Math. Notes **5**, 141–148.

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