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MIXED HADAMARD'S THEOREMS

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Dedicated to Professor Hisaharu Umegaki on his sixtieth birthday and in celebration of his having been honoured as an emeritus Professor of Tokyo Institute of Technology

ABSTRACT. An operator T means a bounded linear operator on a complex Hilbert space H. We give two types of mixed Hadamard's theorems containing the terms T, $|T|$ and $|T^*|$ as extensions of Hadamard's theorem and mixed Schwarz's inequality $|(Tx,y)|^2 \leq (|T|x,x)(|T^*|y,y)$ for any T and for any x and y in H. Also we scrutinize the cases when the equalities in these mixed Hadamard's theorems hold.

1. Statement of the results.

THEOREM 1 (MIXED HADAMARD'S TYPE 1). *For any operator T on H and any $x_1, x_2, \ldots, x_n$ in H, let G_n be defined by*

$$G_n = \begin{vmatrix} (|T|x_1,x_1) & (Tx_1,x_2) & (Tx_1,x_3) & \cdots & (Tx_1,x_n) \\ (T^*x_2,x_1) & (|T^*|x_2,x_2) & (|T^*|x_2,x_3) & \cdots & (|T^*|x_2,x_n) \\ (T^*x_3,x_1) & (|T^*|x_3,x_2) & (|T^*|x_3,x_3) & \cdots & (|T^*|x_3,x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (T^*x_n,x_1) & (|T^*|x_n,x_2) & (|T^*|x_n,x_3) & \cdots & (|T^*|x_n,x_n) \end{vmatrix}.$$

Then

$$0 \leq G_n \leq (|T|x_1,x_1)\prod_{j=2}^{n}(|T^*|x_j,x_j)$$

and $G_n = 0$ if and only if $S_1 = \{|T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}$ is a system of linearly dependent vectors if and only if $S_2 = \{Tx_1, |T^*|x_2, |T^*|x_3, \ldots, |T^*|x_n\}$ is a system of linearly dependent vectors. On the right-hand side, equality holds if and only if $(Tx_1,x_j) = 0$ for $j = 2,3,\ldots,n$ and $(|T^*|x_j,x_k) = 0$ for $j < k$ $(j = 2,3,\ldots,n-1)$ or S_1 contains the zero vector (equivalently, S_2 contains the zero vector).

THEOREM 2 (MIXED HADAMARD'S TYPE 2). *For any operator T on H and any $x_1, x_2, \ldots, x_n$ in H, let G_{2n} be defined by*

$$G_{2n} = \begin{vmatrix} (|T|x_1,x_1) & (Tx_1,x_2) & (|T|x_1,x_3) & (Tx_1,x_4)\cdots(|T|x_1,x_{2n-1}) & (Tx_1,x_{2n}) \\ (T^*x_2,x_1) & (|T^*|x_2,x_2) & (T^*x_2,x_3) & (|T^*|x_2,x_4)\cdots(T^*x_2,x_{2n-1}) & (|T^*|x_2,x_{2n}) \\ (|T|x_3,x_1) & (Tx_3,x_2) & (|T|x_3,x_3) & (Tx_3,x_4)\cdots(|T|x_3,x_{2n-1}) & (Tx_3,x_{2n}) \\ (T^*x_4,x_1) & (|T^*|x_4,x_2) & (T^*x_4,x_3) & (|T^*|x_4,x_4)\cdots(T^*x_4,x_{2n-1}) & (|T^*|x_4,x_{2n}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (|T|x_{2n-1},x_1) & (Tx_{2n-1},x_2)\cdots & & \cdots(|T|x_{2n-1},x_{2n-1}) & (Tx_{2n-1},x_{2n}) \\ (T^*x_{2n},x_1) & (|T^*|x_{2n},x_2)\cdots & & \cdots(T^*x_{2n},x_{2n-1}) & (|T^*|x_{2n},x_{2n}) \end{vmatrix}.$$

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Then

$$0 \le G_{2n} \le \prod_{j=1}^{2n-1} (|T|x_j, x_j)(|T^*|x_{j+1}, x_{j+1})$$

and $G_{2n} = 0$ *if and only if* $S_1 = \{|T|x_1, T^*x_2, |T|x_3, T^*x_4, \ldots, |T|x_{2n-1}, T^*x_{2n}\}$ *is a system of linearly dependent vectors if and only if*

$$S_2 = \{Tx_1, |T^*|x_2, Tx_3, |T^*|x_4, \ldots, Tx_{2n-1}, |T^*|x_{2n}\}$$

is a system of linearly dependent vectors. On the right-hand side, equality holds if and only if $(|T^*|x_{2j}, x_{2k}) = 0$ *for* $j \neq k$, $(|T|x_{2j-1}, x_{2k-1}) = 0$ *for* $j \neq k$, *and* $(Tx_{2j-1}, x_{2k}) = 0$ *for* $j, k = 1, 2, \ldots, n$, *or* S_1 *contains the zero vector* (*equivalently,* S_2 *contains the zero vector*).

COROLLARY 1 (MIXED SCHWARZ'S INEQUALITY). *For any operator* T *and any* x, y *in* H, *then*

$$|(Tx, y)|^2 \le (|T|x, x)(|T^*|y, y).$$

The equality holds if and only if $|T|x$ *and* T^*y *are linearly dependent if and only if* Tx *and* $|T^*|y$ *are linearly dependent.*

REMARK. We would like to emphasize that the equality holds if and only if $|T|x$ and T^*y are linearly dependent if and only if Tx and $|T^*|y$ are linearly dependent. One might believe that the equality would hold if and only if $|T|x$ and $|T^*|y$ are linearly dependent. But here we can give a simple counterexample as follows. Let

$$T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Then

$$|T^*|y = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2|T|x,$$

that is, $|T|x$ and $|T^*|y$ are linearly dependent, but

$$|(Tx, y)|^2 = 36 \neq (|T|x, x)(|T^*|y, y) = 54.$$

This mixed Schwarz's inequality is discussed in [**3**, Problem 138] except the case when the equality holds.

2. Proofs of the results.

In order to show the results, we need the following

THEOREM A. *For* $x_1, x_2, \ldots, x_n$ *in* H, *let* G_n *be the determinant of a square matrix of order* n *defined by* $G_n = |((x_j, x_k))|$. *Then*

$$0 \le G_n \le ||x_1||^2 ||x_2||^2 \cdots ||x_n||^2.$$

On the left-hand side, equality holds if and only if $x_1, x_2, \ldots, x_n$ *are linearly dependent. On the right-hand side, equality holds if and only if* $x_1, x_2, \ldots, x_n$ *are mutually orthogonal or* $\{x_1, x_2, \ldots, x_n\}$ *contains the zero vector.*

The right-hand-side inequality in Theorem A is Hadamard's theorem and also the left-hand-side inequality in Theorem A is well known and can be considered as a generalization of Schwarz's inequality. Many ingenious and elegant proofs of Hadamard's theorem have been given by many authors (for example [**1, 2, 4, 5**]).

THEOREM B. *Let* $T = U|T|$ *be the polar decomposition of* T *where* U *means the partial isometry and* $|T| = (T^*T)^{1/2}$ *with* $N(U) = N(|T|)$ *where* $N(S)$ *denotes the kernel of an operator* S. *Then*

(i) $|T^*| = U|T|U^* = UT^*$;

(ii) $T^* = U^*|T^*|$ *is also the polar decomposition of* T^* *with* $N(U^*) = N(|T^*|)$.

PROOF OF THEOREM B. Theorem B is well known, but the for the sake of convenience, we cite the proof. (i) As U^*U is the initial projection and $U^*U|T| = |T|$, so that $|T^*|^2 = TT^* = U|T||T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2$, then we have $|T^*| = U|T|U^*$ because $U|T|U^*$ is positive. Therefore $|T^*| = U|T|U^* = UT^*$. (ii) By (i), we have $U^*|T^*| = U^*U|T|U^* = |T|U^* = T^*$ and $U^*x = 0$ if and only if $UU^*x = 0$ if and only if $|T|U^*x = 0$ by $N(U) = N(|T|)$ if and only if $T^*x = 0$ if and only if $TT^*x = 0$ if and only if $|T^*|x = 0$. Then $N(U^*) = N(|T^*|)$ and U^* is also a partial isometry. So the proof of (ii) is complete.

PROOF OF THEOREM 1. In Theorem A, we replace x_1 by $|T|^{1/2}x_1$ and x_k by $|T|^{1/2}U^*x_k$ for $k = 2, 3, \ldots, n$. Then we have the following by Theorem B:

$$(|T|^{1/2}x_1, |T|^{1/2}U^*x_k) = (U|T|x_1, x_k) = (Tx_1, x_k) \quad \text{for } k = 2, 3, \ldots, n,$$
$$(|T|^{1/2}U^*x_j, |T|^{1/2}U^*x_k) = (U|T|U^*x_j, x_k) = (|T^*|x_j, x_k) \quad \text{for } j, k = 2, 3, \ldots, n.$$

By Theorem A and Theorem B, we have

$$\begin{aligned} 0 \le G_n &\le \|\,|T|^{1/2}x_1\|^2\|\,|T|^{1/2}U^*x_2\|^2 \cdots \|\,|T|^{1/2}U^*x_n\|^2 \\ &= (|T|x_1, x_1)(|T^*|x_2, x_2) \cdots (|T^*|x_n, x_n). \end{aligned}$$

$G_n = 0$ if and only if $|T|^{1/2}x_1, |T|^{1/2}U^*x_2, \ldots, |T|^{1/2}U^*x_n$ are linearly dependent (by Theorem A) if and only if $|T|x_1, |T|U^*x_2, \ldots, |T|U^*x_n$ are linearly dependent (by the positivity of $|T|^{1/2}$) if and only if $S_1 = \{|T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}$ is a system of linearly dependent vectors (by Theorem B). Then

$$US_1 = \{U|T|x_1, UT^*x_2, UT^*x_3, \ldots, UT^*x_n\}$$

is a system of linearly dependent vectors if and only if

$$S_2 = \{Tx_1, |T^*|x_2, |T^*|x_3, \ldots, |T^*|x_n\}$$

is a system of linearly dependent vectors (by Theorem B).

Conversely assume that S_2 is a system of linearly dependent vectors. Then $U^*S_2 = \{U^*Tx_1, U^*|T^*|x_2, U^*|T^*|x_3, \ldots, U^*|T^*|x_n\}$ is a system of linearly dependent vectors if and only if $S_1 = \{|T|x_1, T^*x_2, T^*x_3, \ldots, T^*x_n\}$ is a system of linearly dependent vectors by Theorem B, so that S_1 is a system of linearly dependent vectors if and only if S_2 is a system of linearly dependent vectors. The proof of equality for the right-hand side follows from Theorem A and the argument stated above in the first half of the proof. So the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. In Theorem A we replace x_{2k} by $|T|^{1/2}U^*x_{2k}$ for $k = 1, 2, \ldots, n$ and x_{2k-1} by $|T|^{1/2}x_{2k-1}$ for $k = 1, 2, \ldots, n$. Then by Theorem B we have

$$\begin{aligned} (|T|^{1/2}U^*x_{2j}, |T|^{1/2}U^*x_{2k}) &= (U|T|U^*x_{2j}, x_{2k}) \\ &= (|T^*|x_{2j}, x_{2k}) \quad \text{for } j, k = 1, 2, \ldots, n, \\ (|T|^{1/2}x_{2j-1}, |T|^{1/2}U^*x_{2k}) &= (U|T|x_{2j-1}, x_{2k}) \\ &= (Tx_{2j-1}, x_{2k}) \quad \text{for } j, k = 1, 2, \ldots, n. \end{aligned}$$

By Theorem A and Theorem B, we have

$$0 \leq G_{2n} \leq \| |T|^{1/2}x_1\|^2 \| |T|^{1/2}U^*x_2\|^2 \cdots \| |T|^{1/2}x_{2n-1}\|^2 \| |T|^{1/2}U^*x_{2n}\|^2$$
$$= (|T|x_1, x_1)(|T^*|x_2, x_2) \cdots (|T|x_{2n-1}, x_{2n-1})(|T^*|x_{2n}, x_{2n}).$$

Since the proofs of the left-hand side and the right-hand side of the equality are given in the same way as in the proofs of Theorem 1, we omit them.

PROOF OF COROLLARY 1. The proof follows from the inequality in Theorem 1 or Theorem 2.

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