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A NOTE ON REAL INTERPOLATION OF HARDY SPACES IN THE POLYDISK

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ABSTRACT. The atomic decomposition for the Hardy spaces H^p of the product of upper half-spaces is used to characterize Peetre's K-functional for two such spaces.

0. Introduction. We present an application of the atomic decomposition for the Hardy spaces H^p of the product of upper half-spaces. Our main results are the computation of Peetre's K-functional for two such spaces and the fact that these spaces are amenable to the method of real interpolation. We also include some applications to Hardy's inequality (for the Fourier transform) and to fractional integration.

1. The K-functional. In order to simplify the presentation of the results, we work exclusively with the domain $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ and its distinguished boundary $\mathbf{R}^2$. We will use the same definitions and notations as in R. Fefferman and S. Y. A. Chang [**1**, **2**]. Points in $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ are denoted by (y,t), where $y = (y_1, y_2) \in \mathbf{R}^2$ and $t = (t_1, t_2)$, $t_1, t_2 > 0$. We reserve the notation $\varphi(u)$ for an even, real-valued, $C^\infty(\mathbf{R})$ function supported in $[-1, 1]$ such that

$$\int_0^\infty \hat{\varphi}(u)^2 \frac{du}{u} = 1, \quad \text{and} \quad \left(\frac{d}{du}\right)^m \hat{\varphi}(0) = 0,$$

for sufficiently large m to be specified. With this function φ we associate the two-parameter dilation

$$\Phi_t(y) = \frac{\varphi(y_1/t_1)\varphi(y_2/t_2)}{t_1 t_2}, \qquad t_1, t_2 > 0,$$

defined on $\mathbf{R}^2$. For a tempered distribution $f \in S'(\mathbf{R}^2)$ we put $f(y,t) = f * \Phi_t(y)$. Further, if $x = (x_1, x_2) \in \mathbf{R}^2$, $\Gamma(x)$ denotes the product cone $\Gamma(x_1) \times \Gamma(x_2)$, or

$$\Gamma(x) = \{(y,t)\colon |x_1 - y_1| < t_1,\ |x_2 - y_2| < t_2\}.$$

We can now introduce the double S-function of f defined by

$$Sf(x) = \left(\iint_{\Gamma(x)} |f(y,t)|^2 \, dy \, \frac{dt_1\, dt_2}{t_1^2 t_2^2}\right)^{1/2}.$$

It is a known fact that for $m \geq 0$ and $1 < p < +\infty$,

$$\|Sf\|_{L^p(\mathbf{R}^2)} \leq C_p \|f\|_{L^p(\mathbf{R}^2)}.$$

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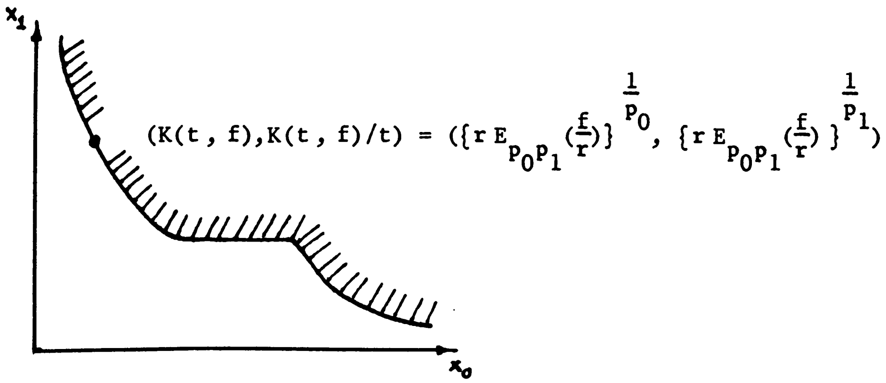

FIGURE 1

It is also well known that we can introduce the Hardy spaces $H^p_\dagger(\mathbf{R}^2 \mathbin{\times_\dagger} \mathbf{R}^2)$ as $\{f \in S'(\mathbf{R}^2)\colon \|Sf\|_{L^p(\mathbf{R}^2)} < +\infty\}$ (modulo a simple normalization). Here $0 < p < +\infty$ and $\|Sf\|_{L^p}$, with $m \sim (1/p - 1)$ for $0 < p \leq 1$, gives one of the possible equivalent H^p "norms" $\|f\|_{H^p}$. In fact, the class H^p is independent of the choice of Φ (see Gundy-Stein [**5**] and Merryfield [**9**]).

Finally, we recall the definition of the K-functional. Given $f \in H^{p_0} + H^{p_1}$ and $t > 0$, put

$$K(t, f; H^{p_0}, H^{p_1}) = \inf_{f=f_0+f_1} \max(\|f_0\|_{H^{p_0}}, t\|f_1\|_{H^{p_1}}).$$

A similar definition holds for two L^p spaces. We have

THEOREM A. *Let* $0 < p_0 < p_1 < +\infty$. *Then*

$$K(t, f; H^{p_0}, H^{p_1}) \approx K(t, Sf; L^{p_0}, L^{p_1}).$$

PROOF. From the subadditivity of the S-function it follows at once that

$$K(t, Sf; L^{p_0}, L^{p_1}) \leq CK(t, f; H^{p_0}, H^{p_1}). \tag{1.1}$$

To prove the converse, we put

$$E_{p_0p_1}(f; H^{p_0}, H^{p_1}) = \inf_{f=f_0+f_1} \max(\|f_0\|_{H^{p_0}}^{p_0}, \|f_1\|_{H^{p_1}}^{p_1}),$$

and similarly for L^{p_0}, L^{p_1}. By looking at the Gagliardo diagram $\Gamma(f) = \{(x_0, x_1) \in \mathbf{R}^2\colon f = f_0 + f_1 \text{ with } \|f_0\|_{H^{p_0}} \leq x_0,\ \|f_1\|_{H^{p_1}} \leq x_1\}$ (see Figure 1), we easily see that the right continuous inverse of $\{E_{p_0p_1}(f/t; H^{p_0}, H^{p_1})\}^{p_0p_1/(p_1-p_0)}$ (as a function of t) is $K(t, f; H^{p_0}, H^{p_1})/t^{p_1/(p_1p_0)}$ and similarly for L^{p_0}, L^{p_1} (cf. Jawerth et al. [**6**]). Now,

$$\begin{aligned} E_{p_0p_1}\left(\frac{f}{c}; L^{p_0}, L^{p_1}\right) &\leq \int_{|f|>1} |f|^{p_0}\, dx + \int_{|f|\leq 1} |f|^{p_1}\, dx \\ &\leq E_{p_0p_1}(cf;\ L^{p_0}, L^{p_1}) \end{aligned}$$

for some constant c independent of f. Hence, to prove the inequality opposite to (1.1), we need to show that

$$E_{p_0p_1}(f; H^{p_0}, H^{p_1}) \leq C\left\{\int_{Sf>1} Sf(x)^{p_0}\, dx + \int_{Sf\leq 1} Sf(x)^{p_1}\, dx\right\}.$$

Next, by the reiteration result of Holmstedt [**1**], it suffices to consider the case $0 < p_0 < 1$, $2 \leq p_1 < +\infty$. It is at this point then the atomic decomposition of Chang and Fefferman [**2**] (also Cohen [**4**]) plays a crucial rôle. For each $k \in \mathbf{Z}_+$ let

$$\begin{aligned}
\Omega_k &= \{Sf > 2^k\}, \\
\mathcal{R}_0 &= \{\text{all dyadic rectangles } R \text{ such that } |R \cap \Omega_0| \leq |R|/2\}, \\
\mathcal{R}_k &= \{\text{all dyadic rectangles } R \text{ such that } |R \cap \Omega_{k-1}| > |R|/2, |R \cap \Omega_k| \leq |R|/2\}, \\
&\qquad k \geq 1.
\end{aligned}$$

For a dyadic rectangle $R = I \times J$, let

$$R^+ = \{(y,t) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ : y \in R,\ |I| < t_1 < 2|I|,\ |J| < t_2 < 2|J|\}.$$

Also $A_k^+ = \bigcup_{R \in \mathcal{R}_k} R^+$. Put

$$f_0(x) = \sum_{k>0} \iint_{A_k^+} f(y,t)\Phi_t(x-y)\, dy\, \frac{dt_1\, dt_2}{t_1 t_2} \equiv \sum_{k>0} b_k(x)$$

and

$$f_1(x) = \iint_{A_0^+} f(y,t)\Phi_t(x-y)\, dy\, \frac{dt_1\, dt_2}{t_1 t_2}.$$

That $f = f_0 + f_1$, in S', follows readily by taking Fourier transforms, since $\int_0^\infty \hat{\varphi}(u)^2\, du/u = 1$. To complete the proof, we consider two claims, namely,

$$\begin{aligned}
\|f_0\|_{H^{p_0}}^{p_0} &\leq C \int_{\{Sf > 1/2\}} Sf(x)^{p_0}\, dx \\
&\leq C \left(\int_{\{Sf>1\}} Sf(x)^{p_0}\, dx + \int_{\{Sf \leq 1\}} Sf(x)^{p_1}\, dx \right)
\end{aligned} \tag{1.2}$$

and

$$\|f_1\|_{H^{p_1}}^{p_1} \leq C \int_{\{Sf \leq 1\}} Sf(x)^{p_1}\, dx. \tag{1.3}$$

The proof of (1.2) is achieved by showing that for each k the normalization of $b_k(x)$ given by $Cb_k(x)/2^k|\Omega_{k-1}|^{1/p_0}$ yields a p_0-atom (in the sense of Chang-Fefferman [**2**]). This would imply that

$$\|f_0\|_{H^{p_0}}^{p_0} \leq C \sum_{k>0} 2^{kp_0} |\Omega_{k-1}| \leq C \int_{\{Sf > 1/2\}} Sf(x)^{p_0}\, dx$$

and thus give (1.2). We need to verify that

(i) $\|b_k\|_{L^2} \le C2^k|\Omega_{k-1}|^{1/2}$,

(ii) $\operatorname{supp} b_k \subseteq \tilde{\Omega}_{k-1} = \bigcup_{R\in\mathcal{R}_k} 4R$,

(iii) $$b_k = \sum_{R\in\mathcal{R}_k} \iint_{R^+} f(y,t)\Phi_t(x-y)\,dy\,\frac{dt_1\,dt_2}{t_1t_2} = \sum_{R\in\mathcal{R}_k} b_{k,R},$$

(iv) $$\int b_{k,R}(x_1,x_2)x_1^m\,dx_1 = \int b_{k,R}(x_1,x_2)x_2^m\,dx_2 = 0$$

for all m up to $\sim (1/p_0 - 1)$,

(v) $\operatorname{supp} b_{k,R} \subseteq 4R$,

(vi) $b_{k,R}$ is in C^1 and $\|b_{k,R}\|_{L^\infty} \le d_R(C2^k|\Omega_{k-1}|^{1/p_0})$,

$$\left\|\frac{\partial}{\partial x_1} b_{k,R}\right\|_{L^\infty} \le d_R\frac{C2^k|\Omega_{k-1}|^{1/p_0}}{|I|^m},$$

$$\left\|\frac{\partial}{\partial x_2} b_{k,R}\right\|_{L^\infty} \le d_R\frac{C2^k|\Omega_{k-1}|^{1/p_0}}{|J|^m}, \qquad R = I\times J,$$

and, finally,

(vii) $$\sum_{R\in\mathcal{R}_k} d_R^2|R| \le C|\Omega_{k-1}|^{1-2/p_0}.$$

The proof of these assertions is similar to that of [**2**, Theorem 1], where it is done for $p_0 = 1$. Let us see that (i) holds. For this purpose pick $g\in L^2$ with $\|g\|_{L^2} = 1$. Then

$$\left|\int b_k(x)g(x)\,dx\right| = \left|\iint_{A_k^+} f(y,t)g(y,t)\,dy\,\frac{dt_1\,dt_2}{t_1t_2}\right|$$
$$\le C\iint_{A_k^+}\left[\frac{1}{t_1t_2}\left|\left\{z\in\mathbf{R}^2\colon M(\chi_{\Omega_{k-1}},z) > \frac{1}{2},\ z\in\Omega_k^c,\ (y,t)\in\Omega(z)\right\}\right|\right]$$
$$\times\,|f(y,t)|\,|g(y,t)|\,dy\,\frac{dt_1\,dt_2}{t_1t_2},$$

where M denotes the strong maximal function. By Tonelli's theorem and Schwartz's inequality this last integral does not exceed

$$\int_{\{M(\chi_{\Omega_{k-1}})>1/2\}\cap\Omega_k^c} Sf(z)Sg(z)\,dz \le \left(\int_{\{M(\chi_{\Omega_{k-1}})>1/2\}\cap\Omega_k^c} Sf(z)^2\,dz\right)^{1/2}\|Sg\|_{L^2}$$
$$\le C2^k|\{M(\chi_{\Omega_{k-1}}) > \tfrac{1}{2}\}|^{1/2}.$$

Hence, by the strong maximal theorem,

$$\left|\int b_k(x)g(x)\,dx\right| \le C2^k|\Omega_{k-1}|^{1/2},$$

which gives (i). The proofs of (ii)–(v) are trivial. As for as for (vi) and (vii), clearly

$$\begin{aligned}\|b_{k,R}\|_{L^\infty} &\le C\iint_{R^+}|f(y,t)|\,dy\,\frac{dt_1\,dt_2}{t_1t_2}/|R|\\ &\le C|R|^{-1/2}\left(\iint_{R^+}|f(y,t)|^2\,dy\,\frac{dt_1\,dt_2}{t_1t_2}\right)^{1/2},\end{aligned}$$

and arguing as above we see that

$$\begin{aligned}\sum_{R\in\mathcal{R}_k}\iint_{R^+}|f(y,t)|^2\,dy\,\frac{dt_1\,dt_2}{t_1t_2} &\le C\int_{\{M(\chi_{\Omega_{k-1}})>1/2\}\cap\Omega_k^c} Sf(z)^2\,dz\\ &\le C2^{2k}|\Omega_{k-1}|.\end{aligned}$$

After the proper normalization this gives (vi)–(vii), except for the estimates of the partial derivatives of $b_{k,R}$, which we leave for the reader to verify.

The proof of (1.3) is similar. For any function h with $\|h\|_{L^{p_1}}=1$ we find, as above,

$$\begin{aligned}\left|\int f_1(x)h(x)\,dx\right| &\le \iint_{A_0^+}|f(y,t)|\,dy\,\frac{dt_1\,dt_2}{t_1t_2}\\ &\le C\int_{\Omega_0^c}Sf(z)Sh(z)\,dz \le C\left(\int_{\Omega_0^c}Sf(z)^{p_1}\,dz\right)^{1/p_1}.\end{aligned}$$

This implies (1.3) and completes the proof.

A few remarks about Theorem A: Let $\mathcal{M}f(x)=\sup_{(y,t)\in\Gamma(x)}|P_t * f(y)|$ be the nontangential maximal function. If we use the proof of the atomic decomposition, due to J. M. Wilson [**10**], which is based on the nontangential maximal function rather than on the square function, an argument entirely analogous to the one above gives

THEOREM A$'$. *Let $0<p_0<p_1<+\infty$. Then*

$$K(t,f;H^{p_0},H^{p_1})\approx K(t,\mathcal{M}f;L^{p_0},L^{p_1}).$$

Moreover, as an immediate consequence of Theorem A (or A$'$) we have the following result of Lin [**8**].

COROLLARY B. *Let $0<p_0<p_1<+\infty$ and $1/p=(1-\theta)/p_0+\theta/p_1$, $0<\theta<1$. Then $(H^{p_0},H^{p_1})_{\theta p}=H^p$ (with equivalent quasinorms).*

Let us mention two simple applications of Corollary B.

In [**6**] it is shown that for any $p_0, 0<p_0<1$, and $f\in H^{p_0}(\mathbf{R}_+^2\times\mathbf{R}_+^2)$,

$$|\hat f(\xi_1,\xi_2)|\le C|\xi_1|^{1/p_0-1}|\xi_2|^{1/p_0-1}\|f\|_{H^{p_0}}.$$

Let $d\mu(\xi_1,\xi_2)=d\xi_1\,d\xi_2/|\xi_1|\,|\xi_2|$, and let $Tf(\xi_1,\xi_2)=|\xi_1|\,|\xi_2|\hat f(\xi_1,\xi_2)$. Then the above estimate means that T is bounded from H^{p_0} to $L^{p_0,\infty}(d\mu)$, $0<p_0<1$. On the other hand, by Plancherel's theorem, T is bounded from H^2 into L^2. Hence, using Corollary B and the Marcinkiewicz interpolation theorem, we obtain

PROPOSITION C (HARDY-LITTLEWOOD IMBEDDING THEOREM). *Let* $0 < p \leq 2$ *and* $f \in H^p(\mathbf{R}_1^2 \times \mathbf{R}_1^2)$. *Then*

$$\left(\iint |\hat{f}(\xi_1, \xi_2)|^p |\xi_1|^{p-2} |\xi_2|^{p-2} \, d\xi_1 \, d\xi_2\right)^{1/p} \leq C\|f\|_{H^p}.$$

In a similar way we can prove results concerning the modified Riesz potential I_α of order α—that is, the operator given by the inverse Fourier transform of

$$(I_\alpha f)\hat{}(\xi_1, \xi_2) = \hat{f}(\xi_1, \xi_2)|\xi_1|^{-\alpha}|\xi_2|^{-\alpha}, \qquad 0 < \alpha < +\infty.$$

PROPOSITION D. *Let* $0 < p < q < +\infty$, $0 < \alpha < 1/p$, *and* $1/q = 1/p - \alpha$. *Then* $\|I_\alpha f\|_{H^q} \leq C\|f\|_{H^p}$.

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