PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 2, February 1986

A NOTE ON INTEGRAL MEANS OF THE DERIVATIVE IN CONFORMAL MAPPING

N. G. MAKAROV

ABSTRACT. There exists a number $p_0 > 1/3$ such that among the derivatives of univalent functions, that of the Koebe function ceases to have the greatest order of growth of L^p-means for all $p \leq p_0$.

Let $\mathbf{D}$ denote the unit disc. We consider the lacunary power series

$$b(z) = \sum_{\nu \geq 1} z^{2^\nu} \quad \text{for } z \in \mathbf{D}.$$

It is well known (see, e.g., [**7**, Chapter 10, §2]) that the primitive

$$h(z) = \int_0^z \exp\left\{-\frac{i}{5} b(z)\right\} dz$$

represents a univalent in $\mathbf{D}$ function. We shall study the growth of the integral means

$$I_p(r, h') = \int_{-\pi}^{\pi} |h'(re^{it})|^p \, dt \qquad (0 \leq r < 1).$$

THEOREM. *There exists a positive number c such that if $0 < p < 1$, then*

$$I_p(r, h') \geq (1-r)^{-cp^2}. \tag{1}$$

In particular, if $p < 1/3 + c/27$, then

$$I_p(r, h') \neq O((1-r)^{1-3p}) \quad \textit{as } r \to 1-.$$

As a motivation, we shall briefly outline the background. Let f be an arbitrary univalent in $\mathbf{D}$ function.

(i) J. Clunie and Ch. Pommerenke [**2**] have proved that for some absolute constant C and for all $p > 0$,

$$I_p(r, f') = O((1-r)^{-Cp^2}) \quad \text{as } r \to 1-. \tag{2}$$

In fact, they have established (2) with $C = 9$, and recently Ch. Pommerenke [**8**] has considerably improved this bound. Our theorem shows that as for the order, the estimate (2) is asymptotically sharp as $p \to 0$.

(ii) For positive p, the distortion theorem implies the trivial estimate

$$I_p(r, f') = O((1-r)^{-3p}) \quad \text{as } r \to 1-.$$

Received by the editors March 1, 1985.
1980 *Mathematics Subject Classification*. Primary 30C55.
Key words and phrases. Univalent functions, integral means of the derivative, lacunary series, ergodic theorem, Hausdorff measures.

On the other hand, the Koebe function $k(z) = z(1-z)^{-1}$, extremal for a large class of problems about integral means, satisfies

$$I_p(r, k') \asymp (1-r)^{1-3p}$$

for $p > 1/3$. J. Feng and T. MacGregor [**4**] have shown that if $p > 2/5$, then

$$I_p(r, f') = O(I_p(r, k')) \quad \text{as } r \to 1- . \tag{3}$$

The latter is certainly false for $p < 1/3$ because there exists a univalent function with the derivative having a radial limit on no set of positive measure. The sharp upper bound for $I_p(r, f')$ seems to be unknown for $p < 2/5$. In particular, it was asked (see, e.g., [**6**, Problem 4.3 and **3**, p. 229]) whether (3) holds for all $p > 1/3$. Our theorem settles this question negatively.

In the proof of the theorem we make use of Hausdorff measures. The α-dimensional Hausdorff measure of a plane set E is denoted by $\Lambda_\alpha(E)$, and $\Lambda_\alpha^\infty(E)$ stands for the infimum of $\sum r_j^\alpha$ taken over all coverings of E with discs of radii r_j. As is known, $\Lambda_\alpha(E) = 0$ iff $\Lambda_\alpha^\infty(E) = 0$.

LEMMA. *Let $\delta < 1/10$. There exists a Borel subset $E_0 \subset \partial\mathbf{D}$ of Hausdorff dimension greater than $1 - 5\delta^2$ such that*

$$\liminf_{r\to 1-} \frac{\operatorname{Im} b(r\varsigma)}{|\log_2(1-r)|} > \frac{1}{3}\delta, \quad \varsigma \in E_0.$$

The lemma is a slight quantitative amplification of a result due to J. Hawkes [**5**, §4].

PROOF OF LEMMA. On the segment $[0,1]$ we define the functions

$$S_n(t) = \sum_{\nu=1}^{n} \sin(2^\nu \cdot 2\pi t).$$

It is easy to verify that if $n = [|\log_2(1-r)|]$, then

$$|\operatorname{Im} b(re^{2\pi t i}) - S_n(t)| \le \text{const.}$$

On $[0,1]$ we also consider the probabilistic measure μ with respect to which the functions $t \mapsto t_\nu$ (= the νth figure in the diadic expansion of t) are independent random variables with the distribution

$$\mu\{t\colon t_\nu = 0\} = \tfrac{1}{2} + \delta, \qquad \mu\{t\colon t_\nu = 1\} = \tfrac{1}{2} - \delta.$$

The measure μ is invariant under the diadic transformation T:

$$T(t) = 2t \qquad (\text{mod } 1),$$

and ergodic with respect to it (see [**1**, Example 3.5]). By the ergodic theorem, for μ-a.e. t

$$\frac{1}{n} S_n(t) = \frac{1}{n}\sum_{\nu=1}^{n} \sin(2\pi T^\nu t) \to \int_0^1 \sin(2\pi t)\, d\mu(t).$$

By the Eagleston-Billingsley theorem [**1**, §14], the measure μ is absolutely continuous with respect to Λ_α provided

$$\alpha < \frac{\text{Ent } T}{\ln 2},$$

where

$$\operatorname{Ent} T = (\tfrac{1}{2}+\delta)|\ln(\tfrac{1}{2}+\delta)| + (\tfrac{1}{2}-\delta)|\ln(\tfrac{1}{2}-\delta)|$$

is the entropy of T. It remains only to note that

$$\begin{aligned} 1 - \frac{\operatorname{Ent} T}{\ln 2} &= \frac{1}{\ln 4}\left[\ln(1-4\delta^2) + 2\delta \ln \frac{1+2\delta}{1-2\delta}\right] \\ &< \frac{1}{\ln 4}\left[-4\delta^2 + \frac{8\delta^2}{1-2\delta}\right] < 5\delta^2 \end{aligned}$$

provided $\delta < 1/10$ and that

$$\begin{aligned} \int_0^1 \sin(2\pi t) d\mu(t) = 2\delta \int_0^{1/2} \sin(2\pi t)\, d\mu(t) &> 2\delta \frac{\sqrt{2}}{2} \mu\left[\frac{1}{8}, \frac{3}{8}\right] \\ &= \frac{\sqrt{2}}{4}\delta(1+2\delta)^2(1-2\delta) > \frac{1}{3}\delta. \end{aligned}$$

PROOF OF THEOREM. Let $\alpha = 1 - 5\delta^2$. Replacing the set E_0 obtained in the Lemma by a suitable subset E of positive α-dimensional Hausdorff measure, we can assume that

$$\operatorname{Im} b(r\varsigma) \geq \tfrac{1}{3}\delta |\log_2(1-r)|$$

for $\varsigma \in E$ and $r \geq r_0$. By $E_{(r)}$ we denote the $(1-r)$-neighbourhood of E. Since

$$|b'(z)| \leq \operatorname{const}(1-|z|)^{-1}$$

we have the estimate

$$\operatorname{Im} b(r\varsigma) \geq \tfrac{1}{3}\delta|\log_2(1-r)| - \operatorname{const}, \qquad \varsigma \in E_{(r)},$$

which implies

$$|h'(r\varsigma)| = \exp\left\{\frac{1}{5}\operatorname{Im} b(r\varsigma)\right\} \geq \operatorname{const}(1-r)^{-\delta/12}, \qquad \varsigma \in E_{(r)}. \tag{4}$$

Let $\lambda = \Lambda_\alpha^\infty(E)$. Then

$$|E_{(r)}| \geq \frac{\lambda}{2}(1-r)^{1-\alpha}. \tag{5}$$

In fact, $E_{(r)}$ consists of disjoint open intervals of length at least $1-r$. Subdivide $E_{(r)}$ in a union of N disjoint intervals (not necessarily open) of length $\leq 2(1-r)$ and $\geq (1-r)$. Then

$$\lambda \leq N[2(1-r)]^\alpha \leq 2N(1-r)^\alpha, \qquad N \geq \tfrac{1}{2}\lambda(1-r)^{-\alpha},$$
$$|E_{(r)}| \geq N(1-r) \geq \tfrac{1}{2}\lambda(1-r)^{1-\alpha}.$$

Inequalities (4) and (5) yield

$$\begin{aligned} \int_{-\pi}^{\pi} |h'(re^{it})|^p\, dt &\geq \operatorname{const}|E_{(r)}|(1-r)^{-\delta p/12} \\ &= \operatorname{const}(1-r)^{5\delta^2 - \delta p/12}. \end{aligned}$$

With $\delta = p/120$, maximizing the exponent, we obtain (1).

REFERENCES

1. P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965.
2. J. Clunie and Ch. Pommerenke, *On the coefficients of univalent functions*, Michigan Math. J. **14** (1967), 71–78.
3. P. Duren, *Univalent functions*, Springer-Verlag, Berlin and New York, 1983.
4. J. Feng and T. MacGregor, *Estimates on integral means of the derivative of univalent functions*, J. Analyse Math. **29** (1976), 203–231.
5. J. Hawkes, *Probabilistic behaviour of some lacunary series*, Z. Wahrsch. Verw. Gebiete **51** (1980), 21–33.
6. Ch. Pommerenke (ed.), *Problems in complex function theory*, Bull. London Math. Soc. **4** (1972), 354–366.
7. ———, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
8. ———, *On the integral means of the derivative of a univalent function*, J. London Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, LENINGRAD STATE UNIVERSITY, LENINGRAD, USSR

Current address: LOMI, Fontanka 27, 191011 Leningrad, USSR