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LARGE-TIME BEHAVIOR OF SOLUTIONS TO CERTAIN QUASILINEAR PARABOLIC EQUATIONS IN SEVERAL SPACE DIMENSIONS

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ABSTRACT. We consider the Cauchy problem, $u_t + \operatorname{div} f(u) = \Delta u$ for $x \in \mathbf{R}^n$, $t > 0$ with $u(x,0) = u_0(x)$. For $n = 1$, suppose $f'' > 0$ and $\int |u_0 - \phi|\,dx < \infty$ where ϕ is piecewise constant and $\phi(x) \to u^+$ (u^-) as $x \to +\infty$ $(-\infty)$. A result of Il'in and Oleinik states that if $\phi(x - kt)$ is an entropy solution of $u_t + \operatorname{div} f(u) = 0$, then $u(x,t)$ approaches a traveling wave solution, $\tilde{u}(x - kt)$, as $t \to \infty$, with $\tilde{u}(x) \to u^+$ (u^-) as $x \to +\infty$ $(-\infty)$. We give two examples which show that this result does not hold for $n \geqslant 2$.

This work concerns the asymptotic behavior of the solution to the Cauchy problem,

$$u_t + \operatorname{div} f(u) = \Delta u \quad \text{for } x \in \mathbf{R}^n,\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbf{R}^n, \tag{2}$$

where $f \in C^2(\mathbf{R}; \mathbf{R}^n)$, and $n \geqslant 2$. We assume throughout the paper that u_0 is a bounded function which approaches a piecewise constant state as $|x| \to \infty$ in the sense that $\int_{\mathbf{R}^n} |u_0 - \phi|\,dx < \infty$, where

$$\phi(x) = a \cdot \chi_\Omega(x) + b \cdot \chi_{\mathbf{R}^n \setminus \Omega}(x) \tag{3}$$

with $a \geqslant b$, and Ω is a connected open subset of $\mathbf{R}^n$ (with piecewise smooth boundary if $n \geqslant 2$). We also assume without loss of generality that $f(a) = f(b) = 0$, since the transformation $\bar{x} = x - kt$, $\bar{t} = t$, where $k = [f(a) - f(b)]/(a - b)$, yields an equation of the form (1) which satisfies this condition.

For $n = 1$, a result of Il'in and Oleinik [3] states that if f is strictly convex and

$$\phi(x) = \phi_c(x) \equiv \begin{cases} a, & x < c, \\ b, & x > c, \end{cases}$$

for some $c \in \mathbf{R}$, then $\lim_{t \to \infty} u(x,t) \equiv \tilde{u}(x)$ exists and satisfies

$$\operatorname{div} f(\tilde{u}) = \Delta \tilde{u} \quad \text{in } \mathbf{R}^n, \tag{4}$$

$$\tilde{u}(x) - \phi(x) \to 0 \quad \text{as } |x| \to \infty, \text{ and} \tag{5}$$

$$\int_{\mathbf{R}^n} (\tilde{u} - \phi)\,dx = \int_{\mathbf{R}^n} (u_0 - \phi)\,dx.$$

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Note that any function ϕ of the form (3) satisfies

$$u_t + \operatorname{div} f(u) = 0. \tag{6}$$

Also since f is convex, the functions $\{\phi_c\}$ are precisely those of the form (3) which satisfy the entropy condition (see (E) below). Thus in one space dimension, $\lim_{t \to \infty} u(x, t)$ exists and satisfies (4) and (5) whenever ϕ satisfies (E).

It is easy to see that the converse is also true. That is, if $\lim_{t \to \infty} u(x, t)$ exists and satisfies (4) and (5), there is a function ψ of the form (3) which satisfies the entropy condition and

$$\int_{\mathbf{R}} |\phi - \psi| \, dx < \infty.$$

The entropy condition for solutions of (6) when $n \geqslant 1$ was formulated by Kruzkov and Vol'pert in [**4** and **6**]. When applied to ϕ as in (3), it can be stated as follows:

$$\langle f(c), n(x) \rangle \leqslant 0 \quad \text{for all } c \in (b, a) \tag{E}$$

H^{n-1}-almost everywhere on $\partial\Omega$, where $n(x)$ is the outward-pointing normal to Ω at x.

In this paper we give two examples which demonstrate that Il'in and Oleinik's result on the large-time behavior of solutions to (1) and (2) fails in dimension $n \geqslant 2$. In both examples, f satisfies the strong convexity condition formulated by Conway in [**1**]. (In fact, we take $f = (0, \ldots, 0, F)$ with F strictly convex.) We take $u_0 = \phi$ where ϕ has the form (3) and ϕ satisfies (E).

In our first example, $\tilde{u}(x) \equiv \lim_{t \to \infty} u(x, t)$ exists and satisfies the elliptic equation (4), but does not inherit the asymptotic values of ϕ as $|x| \to \infty$, as in (5). In our second example, $\tilde{u}$ fails to exist.

EXAMPLE 1. Let $f(u) = (0, 0, \ldots, F(u))$ where $F(u)$ is a smooth strictly convex function with $F(0) = F(1) = 0$. Set $\phi(x) = \chi_\Omega(x)$ where

$$\Omega = \{x = (x', x_n) \colon x_n < -|x'|\}.$$

One readily checks that ϕ satisfies condition (E).

We take $u(x, t)$ to be the solution of (1) and (2) with $u_0(x) = \phi(x)$. Such a solution will exist and be unique in the class

$$C([0, \infty); L^1_{\text{loc}}(\mathbf{R}^n)) \cap L^\infty(\mathbf{R}^{n+1}) \cap C^2(\mathbf{R}^n \times (0, \infty))$$

and satisfies $0 \leqslant u \leqslant 1$. This can be seen by considering a sequence of smooth, bounded functions $u_0^N(x) \to \phi(x)$ in $L^1_{\text{loc}}(\mathbf{R}^n)$. Taking the corresponding solutions of the Cauchy problem [**5**, V, Theorem 8.1] and arguing as in [**6**, §17.2], one obtains both existence and uniqueness in the indicated class. We will show that $\lim_{t \to \infty} u(x, t) \equiv 0$.

Consider first for the sake of comparison the function $g_d(x_n, t)$ satisfying

$$D_{x_n x_n} g_d = D_t g_d + D_{x_n}(F(g_d)) \quad \text{for } -\infty < x_n < \infty,\ t > 0, \tag{7}$$

$$g_d(x_n, 0) = \begin{cases} 1, & x_n < d, \\ 0, & x_n > d. \end{cases}$$

From [3] the functions

$$\int_{-\infty}^{-y} (1 - g_d(s,t))\, ds, \qquad \int_{y}^{\infty} g_d(s,t)\, ds$$

exist, are continuous for $y \in \mathbf{R}$ and $t \geqslant 0$, and together with

$$(1 - g_d(-y,t)), \qquad g_d(y,t)$$

tend to zero as $y \to +\infty$ uniformly in t.

Set $V_d(x_n, t) = \int_{x_n}^{\infty} g_d(s,t)\, ds$; this is continuous for $t \geqslant 0$ and satisfies

$$D_{x_n x_n} V_d = -F(-D_{x_n} V_d) + D_t V_d, \qquad V_d(x_n, 0) = (d - x_n)^+. \tag{8}$$

Moreover

$$\tilde{g}_d(x_n) \equiv \lim_{t\to\infty} g_d(x_n,t) \quad \text{and} \quad \tilde{V}_d(x_n) \equiv \lim_{t\to\infty} V_d(x_n,t)$$

both exist and are solutions to (7) and (8); they are uniquely determined respectively by

$$\int_d^{\infty} \tilde{g}_d(s)\, ds + \int_{-\infty}^{d} (\tilde{g}_d(s) - 1)\, ds = 0, \quad \text{and} \quad \tilde{V}_d(x_n) = \int_{x_n}^{\infty} \tilde{g}_d(s)\, ds.$$

Finally the convergence to $\tilde{g}_d$ is uniform in x_n.

From the maximum principle

$$0 \leqslant u(x,t) \leqslant g_0(x_n, t) \quad \text{for } x \in \mathbf{R}^n,\ t \geqslant 0.$$

Thus $U(x,t) = \int_{x_n}^{\infty} u(x', s, t)\, ds$ is well defined and satisfies

$$\Delta U = -F(-D_{x_n} U) + D_t U, \qquad U(x,0) = (-|x'| - x_n)^+.$$

For any $d < 0$, $w_d \equiv U - V_d$ satisfies

$$\Delta w_d = a(x,t,d) D_{x_n} w_d + D_t w_d \quad \text{and} \quad w_d(x,0) \leqslant -d \cdot \chi_{\{|x'| \leqslant -d\}}(x).$$

Again using the maximum principle, $w_d \leqslant h(x',t)$ for $t \geqslant 0$ where

$$\Delta_{x'} h = D_t h \quad \text{for } x' \in \mathbf{R}^{n-1},\ t > 0$$

and

$$h(x',0) = -d \cdot \chi_{\{|x'| \leqslant -d\}}(x'), \qquad x' \in R^{n-1}.$$

Hence

$$\overline{\lim_{t\to\infty}}\, U(x,t) \leqslant \tilde{V}_d(x_n) + \lim_{t\to\infty} h(x',t) = \tilde{V}_d(x_n).$$

Since $g_d(x_n, t) = g_0(x_n - d, t)$, $\tilde{V}_d(x_n) \to 0$ as $d \to -\infty$. From parabolic estimates it follows that

$$u(x,t) = -D_{x_n} U(x,t) \to 0 \quad \text{as } t \to \infty.$$

Example 2. Consider $f(u)$ as above and $u_0(x) = \phi(x) = \chi_\Omega(x)$ where

$$\Omega = \{(x', x_n) \colon x_n < \psi(x')\} \quad \text{with } \psi \in C^1(\mathbf{R}^{n-1}),\ -1 \leqslant \psi \leqslant 0,$$

and

$$\overline{\lim_{r\to\infty}}\, r^{1-n} \int_{|x'| \leqslant r} \psi\, dx' \neq \underline{\lim_{r\to\infty}} \int_{|x'| \leqslant r} \psi\, dx'.$$

Again one can check that (E) is satisfied. We show that $\lim_{t\to\infty} u(x,t)$ cannot exist pointwise almost everywhere on $\mathbf{R}^n$.

Using the previous remarks on g_d we see that

$$g_{-1}(x_n,t) \leqslant u(x,t) \leqslant g_0(x_n,t)$$

and that the function

$$U(x',t) \equiv \int_{-\infty}^{0} (1 - u(x',s,t)\,ds - \int_0^{\infty} u(x',s,t)\,ds$$

is well defined and satisfies

$$\Delta_{x'}U = D_t U \quad \text{for } t > 0,\ x' \in \mathbf{R}^{n-1},$$
$$U(x',0) = -\psi(x') \quad \text{for } x' \in \mathbf{R}^{n-1}.$$

If $\lim_{t\to\infty} u(x,t)$ is well defined, then $\lim_{t\to\infty} U(x',t) \equiv \tilde{U}(x')$ exists. Hence $\tilde{U}$ is a bounded harmonic function and thus a constant. By a result of [**2**] this is true iff

$$\lim_{r\to\infty} r^{1-n} \int_{|x'|\leqslant r} \psi\,dx' \quad \text{exists.}$$

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