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A NOTE ON A LEMMA OF ZO

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ABSTRACT. In this article we prove that a general class of singular integrals on product spaces maps $L\log L$ boundedly to weak L^1. We use this to prove a theorem about maximal functions which generalize the strong maximal function.

Introduction. In [1] Zo proved the following result:

LEMMA. *Suppose that* $\{K_\alpha(x)\}_{\alpha\in\mathscr{A}}$ *is a collection of kernels on* R^n *satisfying*

$$(1)\qquad \int_{R^n}|K_\alpha(x)|\,dx \leqslant C \quad \textit{for all } \alpha\in\mathscr{A}$$

and

$$(2)\qquad \int_{|x|>2|h|}\sup_{\alpha\in\mathscr{A}}|K_\alpha(x+h)-K_\alpha(x)|\,dx \leqslant C \quad \textit{for all } h\neq 0.$$

Then for $T^*f(x)=\sup_{\alpha\in\mathscr{A}}|f*K_\alpha(x)|$, *we have*

$$m\{T^*f(x)>\lambda\}\leqslant (C/\lambda)\|f\|_{L^1(R^n)} \quad \textit{for all } \lambda>0.$$

The idea behind the proof of this result is the Calderón-Zygmund decomposition [2, 3] of an L^1 function. Our purpose here is to extend the lemma of Zo to product spaces, and the machinery used in doing this is given in [4 and 5]—namely, Journé's geometric lemma, together with an atomic decomposition for $L\log^+ L$ functions which closely resembles the atomic decomposition for H^1 functions in the "product setting". In order to carry out the extension we desire, we shall first prove (as is needed in the classical case) a kind of Calderón-Zygmund theorem for product spaces. This is announced in [5], and we shall indicate here the ideas which must be added to those in [5] in order to prove such a theorem. After this is done, we shall proceed to state and prove Zo's Lemma for product spaces.

A Calderón-Zygmund theorem. We shall begin with some notation. Suppose that T is an integral operator with kernel K on R^n. This means that $Tf(x)=\int_{R^n}K(x,y)f(y)\,dy$ for $x\in R^n$. Suppose that T is bounded on $L^2(R^n)$ with operator norm $\|T\|_{L^2,L^2}$ and that K satisfies

$$(*)\qquad \int_{|x-y|>\gamma|y-y'|}|K(x,y)-K(x,y')|\,dx\leqslant C\gamma^{-\delta} \quad \text{for some } \delta>0,$$

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and every $\gamma \geqslant 2$. Then we define the Calderón-Zygmund norm of T, $\|T\|_{\mathrm{CZ}}$, by

$$\|T\|_{\mathrm{CZ}} = \|T\|_{L^2, L^2} + \inf\{C > 0 \,|\, {*} \text{ holds}\}.$$

Now suppose that $K(x_1, y_1, x_2, y_2)$ is a kernel defined for x_1, $y_1 \in R^n$ and x_2, $y_2 \in R^m$. For each fixed x_1 and y_1 we define the integral $\tilde{K}^1(x_1, y_1)$ which acts on functions on R^m as the operator whose kernel (which we also denote $\tilde{K}^1(x_1, y_1)(x_2, y_2)$) is given by $\tilde{K}^1(x_1, y_1)(x_2, y_2) = K(x_1, y_1, x_2, y_2)$. We define $\tilde{K}^2(x_2, y_2)$ similarly. Then we have the following result **[5]**:

THEOREM. *Let $K(x_1, y_1, x_2, y_2)$ be defined for x_1, $y_1 \in R^n$ and x_2, $y_2 \in R^m$, and let*

$$Tf(x_1, x_2) = \iint_{R^n \times R^m} K(x_1, y_1, x_2, y_2) f(y_1, y_2)\, dy_1\, dy_2$$

for $(x_1, x_2) \in R^n \times R^m$. Assume that

(1) *T is bounded on $L^2(R^n \times R^m)$.*

$$(2) \quad \int_{|x_1 - y_1| > \gamma |y_1 - y_1'|} \left\| \tilde{K}^1(x_1, y_1) - \tilde{K}^1(x_1, y_1') \right\|_{\mathrm{CZ}} dx_1 \leqslant C\gamma^{-\delta} \quad \textit{for all } \gamma \geqslant 2,$$

$$(3) \quad \int_{|x_2 - y_2| > \gamma |y_2 - y_2'|} \left\| \tilde{K}^2(x_2, y_2) - \tilde{K}^2(x_2, y_2') \right\|_{\mathrm{CZ}} dx_2 \leqslant C\gamma^{-\delta} \quad \textit{for all } \gamma \geqslant 2.$$

Then for functions $f(x_1, x_2)$ supported in the set $\{(x_1, x_2) |\ |x_i| < 1,\ i = 1, 2\}$ we have the estimate

$$m\{ (x_1, x_2) |\, |x_1| \leqslant 1, |x_2| \leqslant 1, |Tf(x_1, x_2)| > \lambda \} \leqslant (C'/\lambda) \|f\|_{L \log^+ L}.$$

PROOF. Given $f(x_1, x_2)$ supported in $\{|x_1| < 1\} \times \{|x_2| < 1\} = S$, we may write f (in S) as $f(x_1, x_2) = f_1 + f_2(x_1) + f_3(x_2) + f_4(x_1, x_2)$, where f_1 is constant, $\int_{|x_1| < 1} f_2(x_1)\, dx_1 = 0$, $\int_{|x_2| < 1} f_3(x_2)\, dx_2 = 0$, $\int_{|x_1| < 1} f_4(x_1, x_2)\, dx_1 = 0$ for all x_2, $\int_{|x_2| < 1} f_4(x_1, x_2) dx_2 = 0$ for all x_1, and $\|f_i\|_{L \log L} \leqslant C\|f\|_{L \log L}$ for $i \leqslant 4$. To handle f_1, of course, we may use the L^2 boundedness of T. To handle f_2 and f_3, we proceed as follows: Identify $f_2(x_1)$ with a function of x_1 with values in $L^2(dx_2)$ which happen to be constant on $|x_2| < 1$: $\tilde{f}_2(x_1)(x_2) = f_2(x_1)\chi_{|x_2| < 1}(x_2)$. Then $\tilde{f}_2$ is a function in H^1 of the x_1 variable with values in L^2. By the vector valued classical Calderón-Zygmund theory it follows that for $\tilde{T}\tilde{f}_2(x_1) = \int \tilde{K}^1(x_1, y_1)\tilde{f}_2(y_1)\, dy_1$, we have $\tilde{T}\tilde{f}_2 \in L^1$ with values in L^2. Restricting our attention to $\{ |x_2| < 1\}$, we see that $\tilde{T}\tilde{f}_2$ belongs to L^1 with values in $L^1(|x_2| < 1; dx_2)$, and this implies that Tf_2 belongs to $L^1(S)$.

We now handle the main term, which is $T(f_4)$. We assume $\|f_4\|_{L \log L} = 1$. We first obtain an atomic decomposition of f_4, proceeding as in the H^p theory in **[5]**. Here, because the area integral of f_4 (with respect to a function ψ), $S_\psi(f_4)$ is in Weak L^1, applying the same argument as in **[5]** we obtain a decomposition $f_4 = \sum_{k=-\infty}^{\infty} a_k$, where the a_k are atoms with the following properties: There exists a constant

$C > 10^{10}$ so that

1. a_k is supported in an open set Ω_k with $m(\Omega_k) \leqslant 1/C^k$,

2. each a_k can be further decomposed as $\Sigma_R \alpha_{k,R}$, where $\alpha_{k,R}$ is supported in the double of the maximal dyadic subrectangle $R \subseteq \Omega_k$, has mean value 0 in each of the x_1 and x_2 variables separately, and

$$\sum_R \|\alpha_{k,R}\|_{L^2}^2 \leqslant C^k, \qquad \|a_k\|_{L^2}^2 \leqslant C^k.$$

Now given $\alpha > 0$, we wish to prove that

$$m\left\{ (x_1, x_2) \in S \,\big|\, |T(f_4)| > \alpha \right\} \leqslant (C/\alpha)\|f_4\|_{L\log L}(S).$$

To do this, suppose that α satisfies $C^N < \alpha \leqslant C^{N+1}$. Then, since

$$\left\| \sum_{k=-\infty}^{N+1} a_k \right\|_2 \leqslant \sum_{k=-\infty}^{N+1} \|a_k\|_2 \leqslant AC^{N/2} \leqslant A'\alpha^{1/2},$$

for some constants A, A', we have

$$m\left\{ \left| T\left(\sum_{k=-\infty}^{N+1} a_k \right) \right| > \alpha \right\} \leqslant \frac{C'}{\alpha^2} \left\| \sum_{k=-\infty}^{N+1} a_k \right\|_2^2 \leqslant \frac{C''}{\alpha}.$$

To handle $T(\Sigma_{k=N+2}^{\infty} a_k)$, we consider the atom a_k, $k > N+1$, along with its fine structure. a_k is supported in Ω_k, whose measure is at most $1/C^k$ and can be written as $a_k = \Sigma_R \alpha_{k,R}$, where R is a maximal subrectangle of Ω_k. In **[5]** (see also **[4]**) it is shown how to construct rectangles $\hat{R} \supseteq R$ with the properties

$$(*) \qquad m\left(\bigcup \hat{R}\right) \leqslant Am(\Omega_k) \quad \text{and} \quad \sum_R \int_{c(\hat{R})} |T(\alpha_{k,R})|\, dx \leqslant A.$$

If the reader examines the proof of $(*)$, it will be clear that if we dilate each $\hat{R}$ by a factor of $(C/100)^{(k-N)/4(n+m)}$ to produce $\hat{\hat{R}}$, then

$$(**) \qquad m\left(\bigcup \hat{\hat{R}}\right) \leqslant A\left(\frac{C}{100}\right)^{3(k-N)/4} m(\Omega_k) \leqslant AC^{-1/4(k-N)}\frac{1}{\alpha}$$

and

$$\sum_R \int_{c(\hat{\hat{R}})} |T(\alpha_{k,R})|\, dx \leqslant AC^{-\delta'(k-N)} \quad \text{for some } \delta' > 0.$$

(In $(*)$ and $(**)$ R runs through the maximal subrectangles of Ω_k.)

Let us now sum the estimates of $m(\bigcup \hat{\hat{R}})$ over all $k > N$. This tells us that $m(\bigcup_{k>N} \bigcup_{R \subseteq \Omega_k;\, R \text{ maximal}} \hat{\hat{R}}) \leqslant A'/\alpha$.

Summing the second estimate of $(**)$ over $k > N$ produces

$$\int_{c_E} \left| T\left(\sum_{k>N} a_k \right) \right| dx \leqslant A',$$

where $E = \bigcup_{k>N} \bigcup_{R \subseteq \Omega_k;\, R \text{ maximal}} \hat{\hat{R}}$. Applying Chebychev's estimate, we complete the proof that $m\{|T(f_4)| > \alpha\} \leqslant A/\alpha$, since

$$m\left\{ \left| T\left(\sum_{k>N} a_k \right) \right| > \alpha \right\} \leqslant m(E) + \frac{1}{\alpha} \int_{c_E} \left| T\left(\sum_{k>N} a_k \right) \right| dx.$$

A Zo lemma. Now we shall prove a product version of Zo's lemma. To do this, we require some notation. For a kernel $K(x_1, x_2)$ on $R^n \times R^m$ recall that we denote by $\tilde{K}^1(x_1)$ the operator which acts on functions of the x_2 variable by convolving with $K(x_1, \cdot)$. $\tilde{K}^2(x_2)$ is defined similarly. Suppose that we are given a collection of kernels $\{K_\alpha\}_{\alpha \in \mathscr{A}}$ on $R^n \times R^m$. Then we denote by $[\tilde{K}^1(x_1)]^*$ the operator given by

$$[\tilde{K}^1(x_1)]^* f(x_2) = \sup_{\alpha \in \mathscr{A}} |\tilde{K}^1_\alpha(x_1) f(x_2)|.$$

Then we have the following

PRODUCT ZO LEMMA. *Let $\{K_\alpha\}_{\alpha \in \mathscr{A}}$ be kernels on $R^n \times R^m$ satisfying the following*:

(i) *The operator*

$$T^* f(x_1, x_2) = \sup_{\alpha \in \mathscr{A}} |f * K_\alpha(x_1, x_2)|$$

is bounded on $L^2(R^n \times R^m)$.

(ii) *There exists $\delta > 0$ so that*

$$\int_{|x_1| > \gamma |h|} \left\| [\tilde{K}^1(x_1 + h) - \tilde{K}^1(x_1)]^* \right\|_{L^2, L^2} dx_1 \leqslant C\gamma^{-\delta} \quad \forall h \neq 0, \qquad \gamma \geqslant 2,$$

$$\int_{|x_2| > \gamma |k|} \left\| [\tilde{K}^2(x_2 + k) - \tilde{K}^2(x_2)]^* \right\|_{L^2, L^2} dx_2 \leqslant C\gamma^{-\delta} \quad \forall k \neq 0, \qquad \gamma \geqslant 2.$$

(*Here $\| \ \|_{L^2, L^2}$ denotes the norm of an operator on L^2.*)

(iii)

$$\iint_{\substack{|x_1| > \gamma_1 |h| \\ |x_2| > \gamma_2 |k|}} \sup_{\alpha \in \mathscr{A}} \Big| [K_\alpha(x_1 + h, x_2 + k) - K_\alpha(x_1, x_2 + k)] - [K_\alpha(x_1 + h, x_2) - K_\alpha(x_1, x_2)] \Big| \, dx_1 \, dx_2$$

$$\leqslant C\gamma_1^{-\delta}\gamma_2^{-\delta} \quad \text{if } h, k \neq 0, \ \gamma_1, \gamma_2 \geqslant 2.$$

Then the operator T^* satisfies

$$m\{(x_1, x_2) \in R^n \times R^m \,|\, |x_1|, |x_2| < 1, \ T^* f(x_1, x_2) > \alpha\} \leqslant (C/\alpha) \|f\|_{L \log^+ L}$$

whenever f is supported in $S = \{|x_1| < 1\} \times \{|x_2| < 1\}$.

PROOF. The proof consists in repeating essentially word for word the proof of the Calderón-Zygmund theorem given above, only letting T^* play the role that T plays in that theorem. The only difference occurs in the proof of a lemma from [5] referred to there as the "trivial lemma". We prove an analogous lemma below. (This is used to prove $(*)$ and $(**)$ above for T^*.)

LEMMA. *Let $a(x_1, x_2)$ be supported in the product of two cubes $I \times J = R$. Suppose that $\int_I a(x_1, x_2)\, dx_1 = 0 \ \forall x_2 \in J$ and $\int_J a(x_1, x_2)\, dx_2 = 0 \ \forall x_1 \in I$.*

Then if $\tilde{I}_\gamma$ denotes the cube concentric with I whose sides are γ times as long as those of I,

$$\iint_{x_1 \notin \tilde{I}_\gamma} T^* a(x_1, x_2)\, dx_1\, dx_2 \leqslant C\|a\|_{L^2(R)} m(R)^{1/2} \gamma^{-\delta}.$$

PROOF OF LEMMA. Assume R is centered at 0. We estimate

$$\iint_{\substack{x_1 \notin \tilde{I} \\ x_2 \in \tilde{J}_2}} T^*a(x_1, x_2)\, dx_1\, dx_2$$

first as follows: Fix $x_1 \notin \tilde{I}_\gamma$. Then

$$\int_{\tilde{J}_2} T^*a(x_1, x_2)\, dx_2 \leqslant C\left(\int_{\tilde{J}_2} (T^*a)^2(x_1, x_2)\, dx_2\right)^{1/2} m(J)^{1/2}.$$

Now introduce the notation that $a_{x_1}(x_2) = a(x_1, x_2)$ and $T_\alpha f = f * K_\alpha$. We see that

$$(T_\alpha a)_{x_1} = \int_I \tilde{K}^1_\alpha(x_1 - y_1) \cdot a_{y_1}\, dy_1$$

so that, since $\int_I a_{y_1}\, dy_1 = 0$, this equals

$$\int_I \left[\tilde{K}^1_\alpha(x_1 - y_1) - \tilde{K}^1_\alpha(x_1)\right] a_{y_1}\, dy_1.$$

It follows that

$$(T^*a)_{x_1} \leqslant \int_I \left[\tilde{K}^1(x_1 - y_1) - \tilde{K}^1(x_1)\right]^* a_{y_1}\, dy_1$$

and

$$\|(T^*a)_{x_1}\|_{L^2(R^m)} \leqslant \int_I \left\|\left[\tilde{K}^1(x_1 - y_1) - \tilde{K}^1(x_1)\right]^*\right\|_{L^2, L^2} \|a_{y_1}\|_{L^2(R^m)}\, dy_1.$$

Therefore

$$\int_{\tilde{J}_2} T^*a(x_1, x_2)\, dx_2 \leqslant \int_I \left\|\left[\tilde{K}^1(x_1 - y_1) - \tilde{K}^1(x_1)\right]^*\right\|_{L^2, L^2} \|a_{y_1}\|_{L^2} m(J)^{1/2}\, dy_1.$$

Integrating in $x_1 \notin \hat{I}_\gamma$,

$$\begin{aligned}
&\iint_{\substack{x_1 \notin \tilde{I}_\gamma \\ x_2 \in \tilde{J}_2}} T^*a(x_1, x_2)\, dx_1\, dx_2 \\
&\quad \leqslant \iint_{I\, x_1 \notin \tilde{I}_\gamma} \left\|\left[\tilde{K}^1(x_1 - y_1) - \tilde{K}^1(x_1)\right]^*\right\|_{L^2, L^2} \|a_{y_1}\|_{L^2} m(J)^{1/2}\, dx_1\, dy_1 \\
&\quad \leqslant C\gamma^{-\delta} \int_I \|a_{y_1}\|_{L^2} m(J)^{1/2}\, dy_1 \leqslant C\gamma^{-\delta} \|a\|_{L^2(R)} m(R)^{1/2}.
\end{aligned}$$

Now we estimate $\iint_{x_1 \notin \tilde{I}_\gamma;\, x_2 \notin \tilde{J}_2} T^*a\, dx_1\, dx_2$. Now let $x_1 \notin \tilde{I}_\gamma$, $x_2 \notin \tilde{J}_2$.

$$\begin{aligned}
T_\alpha a(x_1, x_2) &= \iint_{I \times J} a(y_1, y_2) K_\alpha(x_1 - y_1, x_2 - y_2)\, dy_1\, dy_2 \\
&= \iint_{I \times J} a(y_1, y_2)\{[K_\alpha(x_1 - y_1, x_2 - y_2) - K_\alpha(x_1, x_2 - y_2)] \\
&\qquad - [K_\alpha(x_1 - y_1, x_2) - K_\alpha(x_1, x_2)]\}\, dy_1\, dy_2.
\end{aligned}$$

Denote the second difference in braces by ΔK_α and $\sup_\alpha |\Delta K_\alpha|$ by $\Delta^* K$. Then

$$\iint_{\substack{x_1 \notin \tilde{I} \\ x_2 \notin \tilde{J}_2}} T^* a \, dx_1 \, dx_2 \leqslant \iint_R |a(y_1, y_2)| \iint_{\substack{x_1 \notin \tilde{I}_\gamma \\ x_2 \notin \tilde{J}_2}} |\Delta^* K(x_1, x_2, y_1, y_2)| \, dx_1 \, dx_2 \, dy_1 \, dy_2$$

$$\leqslant C\gamma^{-\delta} \|a\|_{L^1(R)} \leqslant C\gamma^{-\delta} \|a\|_{L^2(R)} m(R)^{1/2},$$

and this proves the lemma.

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