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ENLARGEMENTS OF ALMOST OPEN MAPPINGS

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ABSTRACT. If the enlargement of a bounded linear operator has dense range, then the operator must be almost open.

Introduction. If X is a normed space we write [**1**]

(0.1) $$\mathbf{Q}(X) = l_\infty(X)/c_0(X)$$

for the *enlargement* of X, and if $T \in \mathrm{BL}(X, Y)$ is a bounded linear operator between normed spaces we write

(0.2) $$\mathbf{Q}(T) \colon \mathbf{Q}(X) \to \mathbf{Q}(Y)$$

for the operator induced by T, so that for each $x \in l_\infty(X)$ we have

(0.3) $$\mathbf{Q}(T)(x + c_0(X)) = (Tx) + c_0(Y).$$

Now we recall that

(0.4) $$\mathbf{Q}(T)\ \textit{one-one} \Rightarrow T\ \textit{bounded below} \Rightarrow \mathbf{Q}(T)\ \textit{bounded below}$$

and [**1**, Theorem 4.1],

(0.5) $$\mathbf{Q}(T)\ \textit{almost open} \Rightarrow T\ \textit{almost open} \Rightarrow \mathbf{Q}(T)\ \textit{open}.$$

It is the purpose of this note to improve (0.5) by confirming the conjecture (4.1.3) of [**1**].

1.1

THEOREM. *If $T \in \mathrm{BL}(X, Y)$ is a bounded linear operator between normed spaces then*

(1.1.1) $$\mathbf{Q}(T)\ \textit{dense} \Rightarrow T\ \textit{almost open} \Rightarrow \mathbf{Q}(T)\ \textit{open}.$$

PROOF. Suppose $\varphi \colon l_\infty \to \mathbf{C}$ is a bounded linear functional for which

(1.1.2) $$c_0 \subseteq \varphi^{-1}(0),$$

then for each normed space X we may define

(1.1.3) $$\hat{\varphi_X} \colon \mathbf{Q}(X^\dagger) \to \mathbf{Q}(X)^\dagger$$

by setting, for each $f \in l_\infty(X^\dagger)$ and each $x \in l_\infty(X)$,

(1.1.4) $$\hat{\varphi_X}([f])([x]) = \varphi(f_\cdot(x_\cdot));$$

here $X^\dagger$ denotes the usual *dual* of the normed space X, $[x] = x + c_0(X)$ and $[f] = f + c_0(X^\dagger)$ are cosets, and $f_\cdot(x_\cdot) = a \in l_\infty$ where $a_n = f_n(x_n)$ for each $n \in \mathbb{N}$. The reader should check, using (1.1.2), that the right-hand side of (1.1.4) depends

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only on the cosets $[f]$ and $[x]$ and that the linear mapping $\hat{\varphi_X}([f])\colon \mathbf{Q}(X)^\dagger \to \mathbb{C}$ is bounded. Using the Hahn-Banach theorem we claim

(1.1.5) $[0] \neq [f] \in \mathbf{Q}(X^\dagger) \Rightarrow \varphi(f_\cdot(x_\cdot)) \neq 0$ *for some* $x \in l_\infty(X)$, $\varphi \in (l_\infty/c_0)^\dagger$;

for if $f \in l_\infty(X^\dagger)$ is not in $c_0(X^\dagger)$, then $\limsup_n \|f_n\| > 0$ and hence there is $x \in l_\infty(X)$ for which $\limsup_n |f_n(x_n)| > 0$, so that $f_\cdot(x_\cdot) \in l_\infty$ is not in c_0. Now by the Hahn-Banach theorem there is $\varphi \in (l_\infty)^\dagger$ satisfying (1.1.2) for which $\varphi(f_\cdot(x_\cdot)) \neq 0$.

If $T \in \mathrm{BL}(X, Y)$ and $\varphi \in (l_\infty)^\dagger$ satisfies (1.1.2) then we claim

$$\mathbf{Q}(T)^\dagger \circ \hat{\varphi_Y} = \hat{\varphi_X} \circ \mathbf{Q}(T^\dagger); \tag{1.1.6}$$

for this is just the associative property

(1.1.7) $\varphi(g_\cdot(Tx)_\cdot) = \varphi((gT)_\cdot(x_\cdot))$ *for each* $x \in l_\infty(X)$, $g \in l_\infty(Y^\dagger)$.

We are ready to make our final claim: if $T \in \mathrm{BL}(X, Y)$ then

$$\mathbf{Q}(T)^\dagger \text{ one-one} \Rightarrow \mathbf{Q}(T^\dagger) \text{ one-one}. \tag{1.1.8}$$

Indeed suppose $\mathbf{Q}(T^\dagger)$ is not one-one, so that there is $g \in l_\infty(Y^\dagger)$ for which

$$gT \in c_0(X^\dagger) \quad \text{and} \quad g \notin c_0(Y^\dagger), \tag{1.1.9}$$

and then by (1.1.5) there is $\varphi \in (l_\infty)^\dagger$ and $y \in l_\infty(Y)$ for which

$$c_0 \subseteq \varphi^{-1}(0) \quad \text{and} \quad \varphi(g_\cdot(y_\cdot)) \neq 0; \tag{1.1.10}$$

but now

$$\mathbf{Q}(T)^\dagger(\hat{\varphi_Y}[g]) = [0] \in \mathbf{Q}(X)^\dagger \quad \text{and} \quad [0] \neq \hat{\varphi_Y}[g] \in \mathbf{Q}(Y)^\dagger. \tag{1.1.11}$$

A familiar application of the Hahn-Banach separation theorem now gives (1.1.1): If $T \in \mathrm{BL}(X, Y)$ then

$$\mathbf{Q}(T) \text{ dense} \Rightarrow \mathbf{Q}(T)^\dagger \text{ one-one} \Rightarrow \mathbf{Q}(T^\dagger) \text{ one-one} \tag{1.1.12}$$

and

$$\mathbf{Q}(T^\dagger) \text{ one-one} \Rightarrow T^\dagger \text{ bounded below} \Rightarrow T \text{ almost open}. \tag{1.1.13}$$

For an alternative proof of Theorem 1.1 we can use ultrafilters on $\mathbb{N}$ instead of linear functionals on l_∞/c_0 **[2]**.

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