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COMPARISON THEOREMS FOR SECOND ORDER DIFFERENTIAL SYSTEMS

W. J. KIM

ABSTRACT. Comparison theorems are proved for second order linear differential systems of the form $(R_i y')' + P_i y = 0$, where R_i and P_i are continuous $n \times n$ matrices and R_i is invertible, $i = 1, 2$.

Let R and P be $n \times n$ matrices with real elements which are continuous and let R be invertible on an x-interval $[a, \omega)$. We shall consider the second-order vector differential equation

$$\text{(E)} \qquad (R(x)y')' + P(x)y = 0.$$

If (E) has a nontrivial solution v satisfying $v(b) = v'(c) = 0$ $[v'(b) = v(c) = 0]$ for some b and c, $a \leqslant b < c < \omega$, we define $\eta(b)$ $[\phi(b)]$ to be the infimum of ξ, $b \leqslant \xi < \omega$, such that there exists a nontrivial solution u of (E) satisfying $u(b) = u'(\xi) = 0$ $[u'(b) = u(\xi) = 0]$. Otherwise, we put $\eta(b) = \omega$ $[\phi(b) = \omega]$. If $\eta(b) < \omega$ $[\phi(b) < \omega]$, then (E) has a nontrivial solution y such that $y(b) = y'(\eta(b)) = 0$ $[y'(b) = y(\phi(b)) = 0]$. $\phi(b)$ is called the *right-hand focal point* of b. In recent years some authors have referred to $\eta(b)$ as a focal point of b; however, this appears to be inconsistent with the long-term usage of "focal" [**13**]. In Picone's terminology, $\eta(b)$ is a right-hand pseudoconjugate of b and $\phi(b)$ is a right-hand hemiconjugate to b. We shall henceforth call $\eta(b)$ the *right-hand pseudoconjugate* of b.

Morse [**11**] was the first to obtain generalizations of the classical Sturm separation and comparison theorems for the second-order vector differential equations

$$(\text{E}_i) \qquad (R_i(x)y')' + P_i(x)y = 0, \qquad i = 1, 2,$$

where R_i and P_i are $n \times n$ matrices with continuous and real elements and R_i is invertible on $[a, \omega)$, $i = 1, 2$. Other comparison results of a different nature have been recently proved by Ahmad and Lazer [**1–3**] for the case $R_i = I$, and also by others [**6, 10, 14, 15**] under various assumptions on R_i and P_i. In [**15**] Tomastik also considered comparison theorems for the right-hand and left-hand focal points. It is to be noted that in most of these studies the case $P_1 \equiv P_2$ is specifically excluded.

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Let $\eta_i(b)$ $[\phi_i(b)]$ be the right-hand pseudoconjugate [the right-hand focal point] of b for (E_i), $i = 1, 2$. The purpose of this paper is to present theorems comparing $\eta_1(b)$ $[\phi_1(b)]$ and $\eta_2(c)$ $[\phi_2(c)]$, where b and c are not necessarily equal. For the special case $R_1 \equiv R_2$, $P_1 \equiv P_2$, these results become "separation theorems," from which we can further deduce that $\eta_i(x)$ $[\phi_i(x)]$ is a nondecreasing function of x.

The Riccati equation technique [**5**, **8**, **9**, **12**] adapted to the second-order system (E) is used to establish the main theorems.

THEOREM 1. *Let b be a point on the interval $[a, \omega)$. Every nontrivial vector solution y of* (E) *with $y(b) = 0$ has the property that $y'(x) \neq 0$, $b \leqslant x < \omega$, if and only if the matrix Riccati system*

$$\text{(MR)} \qquad S' = R^{-1} + SPS, \qquad S(b) = 0,$$

has a solution on $[b, \omega)$.

PROOF. Let Y be the solution of the matrix system

$$\text{(M)} \qquad (R(x)Y')' + P(x)Y = 0, \qquad Y(b) = 0, \quad Y'(b) = I.$$

To prove the necessity, let α be an arbitrary nonzero constant vector. Then $y(x) \equiv Y(x)\alpha$ is a nontrivial solution of (E) with $y(b) = 0$. Since $y'(x) = Y'(x)\alpha \neq 0$, $b \leqslant x < \omega$, we see that the determinant of $Y'(x)$ does not vanish on $[b, \omega)$. Thus, Y' is invertible on $[b, \omega)$. Since R is also invertible on $[b, \omega)$, so is RY'. Consequently, $S \equiv Y(RY')^{-1}$ is defined and continuously differentiable on $[b, \omega)$ and $S(b) = 0$. Differentiating S, we obtain $S' = R^{-1} + SPS$, which proves that S is a solution of (MR) on $[b, \omega)$.

For $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$, i, $j = 1, \ldots, n$, and we define

$$\int_b^x A(t)\,dt = \left(\int_b^x a_{ij}(t)\,dt\right).$$

In order to prove the sufficiency, we require the following lemma.

LEMMA 1. *The matrix Riccati equation* (MR) *has a unique solution S on $J = [b, \eta(b))$. The solution S is continuously differentiable and nontrivial; furthermore, it is nonnegative on J if*

$$\text{(1)} \qquad R^{-1}(x) \geqslant 0, \qquad P(x) \geqslant 0, \qquad b \leqslant x < \eta(b).$$

PROOF. If $R^{-1} = (t_{ij})$, $P = (p_{ij})$, and $S = (s_{ij})$, the system (MR) is equivalent to the system of n^2 first-order equations

$$s'_{ij} = t_{ij} + \sum_{k=1}^{n} s_{ik} \sum_{l=1}^{n} p_{kl} s_{lj}, \qquad s_{ij}(b) = 0,$$

i, $j = 1, 2, \ldots, n$. Evidently, the above system may be cast into a vector equation of the form

$$\text{(2)} \qquad s' = f(x, s), \qquad s(b) = 0,$$

where s and f are n^2-dimensional vectors. The vector-valued function f is continuous on $D = \{(x, s)\colon x \in J,\ |s| < \infty\}$; indeed, it is continuously differentiable on D as a function of s. Therefore, $f(x, s)$ satisfies a Lipschitz condition with respect to s on any compact and convex subset of D (see, e.g., [**4**, p. 142]) and there exists a unique solution $s \in C'$ of (2) on some interval $[b, c]$, $b < c < \eta(b)$ [7, p. 10]. Hence, the matrix Riccati system (MR) has a unique solution S, continuously differentiable on the interval $[b, c]$. Let $c_1 = \sup\{c\colon$ (MR) has a unique solution on $[b, c]$, $b < c < \eta(b)\}$, and let S^* be the unique solution of (MR) on $[b, c_1)$. Since the derivative of every nontrivial vector solution y of (E) with $y(b) = 0$ does not vanish on $[b, \eta(b))$, it follows from the necessity of Theorem 1 that (MR) has a solution, say S°, on $[b, \eta(b))$; thus, $S^* = S^\circ$ on $[b, c_1)$ by the uniqueness of solutions. If $c_1 < \eta(b)$, we may assume that S^* is defined on $[b, c_1]$ (by setting $S^*(c_1) = \lim_{x \to c_1^-} S^*(x) = S^\circ(c_1)$, if necessary). The solution S^* may then be continued to a right neighborhood $[c_1, c_1 + \varepsilon]$, $\varepsilon > 0$, of c_1 [7, p. 15]. This implies that (MR) has a unique solution on $[b, c_1 + \varepsilon]$, $\varepsilon > 0$, contrary to the choice of c_1. Therefore, $c_1 = \eta(b)$ and (MR) has a unique solution S on J.

The solution S is continuously differentiable because R^{-1} and P are continuous. Furthermore, S is nontrivial because $R^{-1} \not\equiv 0$—R^{-1} is invertible on $[b, \eta(b))$ and it cannot have zero rows or zero columns at any point of $[b, \eta(b))$—and it may be obtained as the uniform limit of the successive approximations $\{S_i\}$ defined recursively by the formula

$$S_0(x) = 0,$$
$$S_{k+1}(x) = \int_b^x R^{-1}(t)\,dt + \int_b^x S_k(t)P(t)S_k(t)\,dt,$$

$k = 0, 1, \ldots,$ on some interval $[b, d]$, $b < d < \eta(b)$ (see, e.g., [7, p. 12]). Due to the inequalities (1), $S_k \geqslant 0$ on $[b, d]$, $k = 0, 1, \ldots,$ and therefore the uniform limit $S \geqslant 0$ on $[b, d]$. Let $d_1 = \sup\{d\colon S \geqslant 0$ on $[b, d]$, $b < d < \eta(b)\}$. Then $S \geqslant 0$ on $[b, d_1)$. We shall prove that $d_1 = \eta(b)$. If $d_1 < \eta(b)$, then $0 \leqslant S < \infty$ on $[b, d_1]$ by the continuity of S. In this case, S may again be represented on some interval $[d_1, e]$, $d_1 < e < \eta(b)$, as the uniform limit of the successive approximations

$$S_0(x) = S(d_1) \geqslant 0,$$
$$S_{k+1}(x) = S(d_1) + \int_{d_1}^x R^{-1}(t)\,dt + \int_{d_1}^x S_k(t)P(t)S_k(t)\,dt,$$

$k = 0, 1, \ldots.$ Since $S_k \geqslant 0$ on $[d_1, e]$, $k = 0, 1, \ldots, S \geqslant 0$ on $[d_1, e]$. We are thus led to the conclusion that $S \geqslant 0$ on $[b, e]$, contrary to the choice of d_1. Consequently, $d_1 = \eta(b)$ and $S \geqslant 0$ on $[b, \eta(b))$.

Returning now to the proof of Theorem 1, we shall first prove that $|Y'|$, the determinant of Y', does not vanish on $[b, \omega)$ if (MR) has a solution S on $[b, \omega)$. Since $|Y'|$ is continuous and $|Y'(b)| = 1$ by (M), $|Y'|$ does not vanish on some right neighborhood N of the point b, that is, Y' is invertible on N. Since R is invertible, $Y(RY')^{-1}$ is defined on N and satisfies (MR), as was shown earlier. Due to the uniqueness of solutions of the initial value problem (MR) proved in Lemma 1, we have $S = Y(RY')^{-1}$ on N. Suppose that $|Y'|$ vanishes at some point on $[b, \omega)$: Let $\tilde{x}$ be the first point to the right of b at which $|Y'|$ vanishes. Then there exists a nonzero constant vector β such that $Y'(\tilde{x})\beta = 0$. On the interval $[b, \tilde{x})$ we have $S = Y(RY')^{-1}$, which may be written as $SRY' = Y$; this equality is indeed valid on $[b, \tilde{x}]$ because S, R, Y and Y' are continuous on $[b, \tilde{x}]$. In particular, $S(\tilde{x})R(\tilde{x})Y'(\tilde{x})\beta = Y(\tilde{x})\beta = 0$. But this is absurd since $w \equiv Y\beta$ is a nontrivial solution of (E) and it cannot satisfy the condition $w(\tilde{x}) = w'(\tilde{x}) = 0$. Therefore, $|Y'|$ cannot vanish on $[b, \omega)$.

If y is any nontrivial solution of (E) with $y(b) = 0$, then there exists a nonzereo constant vector γ such that $y = Y\gamma$. Evidently, $y' = Y'\gamma \neq 0$ on $[b, \omega)$ because $|Y'| \neq 0$ on $[b, \omega)$. This completes the proof.

Another result we need for proving comparision theorems is a version of Lemma 3.2 [**12**], strengthened for the matrix Riccati systems

$$(\mathrm{MR}_i) \qquad S' = R_i^{-1} + SP_iS, \qquad S(b) = 0, \qquad i = 1, 2.$$

LEMMA 2. *Let R_i and P_i be $n \times n$ matrices with continuous and real elements and let R_i be invertible on an interval $[a, \omega)$, $i = 1, 2$. Assume that*

$$(3) \quad 0 \leqslant \int_b^x R_2^{-1}(t)\,dt \leqslant \int_b^x R_1^{-1}(t)\,dt, \qquad 0 \leqslant P_2(x) \leqslant P_1(x), \quad b \leqslant x < \omega,$$

for some b, $a \leqslant b < \omega$. If there exists a nonnegative differentiable matrix S defined on $[b, \omega)$ satisfying the matrix inequality

$$(4) \qquad S' \geqslant R_1^{-1} + SP_1S, \qquad S(b) = S_b \geqslant 0,$$

then the matrix differential equation

$$(5) \qquad T' = R_2^{-1} + TP_2T, \qquad T(b) = T_b, \quad S_b \geqslant T_b \geqslant 0,$$

has a continuous solution $T \leqslant S$ on $[b, \omega)$.

PROOF. The existence of T is proved by the iteration procedure

$$(6) \quad T_0(x) = S, \qquad T_{k+1}(x) = T_b + \int_b^x R_2^{-1}(t)\,dt + \int_b^x T_k(t)P_2(t)T_k(t)\,dt,$$

$b \leqslant x \leqslant \omega$, $k = 0, 1, \ldots$ (cf. [**12**]). For $k = 0$,

$$\begin{aligned} 0 \leqslant T_1(x) &= T_b + \int_b^x R_2^{-1}(t)\,dt + \int_b^x S(t)P_2(t)S(t)\,dt \\ &\leqslant S_b + \int_b^x R_1^{-1}(t)\,dt + \int_b^x S(t)P_1(t)S(t)\,dt \leqslant S(x) = T_0(x), \end{aligned}$$

due to (3), (4), (5) and the nonnegativity of S; hence, T_1 is continuously differentiable and $0 \leqslant T_1 \leqslant T_0$ on $[b, \omega)$. From (6) we see that $T_{k+1} \geqslant 0$ if $T_k \geqslant 0$. Also, for $k = 0, 1, \ldots,$

$$T_{k+1}(x) - T_k(x) = \int_b^x \left[T_k(t)P_2(t)T_k(t) - T_{k-1}(t)P_2(t)T_{k-1}(t)\right] dt,$$

where the integrand is nonpositive if $0 \leqslant T_k \leqslant T_{k-1}$. Therefore, $0 \leqslant T_{k+1} \leqslant T_k$ if $0 \leqslant T_k \leqslant T_{k-1}$. Since $0 \leqslant T_1 \leqslant T_0$, the sequence of continuously differentiable matrices $\{T_k\}$ decreases monotonically and is bounded below by zero. Furthermore, the sequence is equicontinuous on any compact subinterval K of $[b, \omega)$. To show this, let $\|A\|$ be the norm of an $n \times n$ matrix $A = (a_{ij})$ defined by $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$. Let $M > 0$ be a constant such that $\|R_2^{-1}\|$, $\|P_2\|$, and $\|T_k\|$, $k = 0, 1, \ldots,$ are all bounded by M on K. From (6),

$$T_{k+1}(x_2) - T_{k+1}(x_1) = \int_{x_1}^{x_2} R_2^{-1}(t)\,dt + \int_{x_1}^{x_2} T_k(t)P_2(t)T_k(t)\,dt,$$

$x_1, x_2 \in K$, $k = 0, 1, \ldots$. Thus,

$$\|T_{k+1}(x_2) - T_{k+1}(x_1)\| \leqslant \int_{x_1}^{x_2} \|R^{-1}(t)\|\,|dt| + \int_{x_1}^{x_2} \|T_k(t)P_2(t)T_k(t)\|\,|dt|$$
$$\leqslant (M + M^3)|x_2 - x_1|, \qquad x_1, x_2 \in K,$$

$k = 0, 1, \ldots,$ and this implies that the sequence $\{T_k\}$ is equicontinuous on K. Since it is also uniformly bounded on K, $\{T_k\}$ converges uniformly on K. The uniform limit $T = \lim_{k \to \infty} T_k$ is a continuous solution of (5) and $T \leqslant T_0 = S$ on K. Since this conclusion holds for every compact subinterval of $[b, \omega)$, it holds for $[b, \omega)$.

We are now ready to prove a comparison theorem for $\eta_i(x)$, the right-hand pseudoconjugate function of (E_i), $i = 1, 2$, defined on $[a, \omega)$.

THEOREM 2. *Let b be a point on the interval $[a, \omega)$. If*

$$(7) \quad R_1^{-1}(x) \geqslant 0, \qquad \int_b^x R_1^{-1}(t)\,dt \geqslant \int_b^x R_2^{-1}(t)\,dt \geqslant 0, \qquad P_1(x) \geqslant P_2(x) \geqslant 0,$$

$b \leqslant x < \omega$, *then* $\eta_1(b) \leqslant \eta_2(b)$. *If the stronger condition*

$$(8) \qquad R_1^{-1}(x) \geqslant R_2^{-1}(x) \geqslant 0, \qquad P_1(x) \geqslant P_2(x) \geqslant 0,$$

$a \leqslant x < \omega$, *holds, then* $\eta_1(b) \leqslant \eta_2(c)$, $a \leqslant b \leqslant c < \omega$.

PROOF. Every nontrivial solution y of (E_1) with $y(b) = 0$ has the property that $y' \neq 0$ on $[b, \eta_1(b))$. Hence, the corresponding matrix Riccati equation (MR_1) has a solution S on $[b, \eta_1(b))$ by Theorem 1. The solution S is nontrivial and nonnegative on $[b, \eta_1(b))$ by Lemma 1. According to (7) and Lemma 2, the matrix Riccati system (MR_2) associated with (E_2) has a continuous solution T on $[b, \eta_1(b))$. Therefore, by Theorem 1, every nontrivial solution vector w of (E_2) with $w(b) = 0$ has the property that $w' \neq 0$ on $[b, \eta_1(b))$; consequently, $\eta_1(b) \leqslant \eta_2(b)$.

If (8) holds and c is an arbitrary point of $[a, \omega)$, then

$$\int_c^x R_1^{-1}(t)\,dt \geqslant \int_c^x R_2^{-1}(t)\,dt \geqslant 0, \qquad a \leqslant c \leqslant x < \omega.$$

For $a \leqslant b \leqslant c < \eta_1(b)$, (MR_1) has a nontrivial solution S which is continuous and nonnegative on $[b, \eta_1(b))$ by Theorem 1 and Lemma 1. Applying Lemma 2 to the interval $[c, \eta_1(b))$, we conclude that the system $T' = R_2^{-1} + TP_2T$, $T(c) = 0$, has a matrix solution T on $[c, \eta_1(b))$. Again by Theorem 1, if v is any nontrivial solution of (E_2) with $v(c) = 0$, then v' does not vanish on $[c, \eta_1(b))$. Therefore, $\eta_1(b) \leqslant \eta_2(c)$, $a \leqslant b \leqslant c < \eta_1(b)$.

If, on the other hand, $\eta_1(b) \leqslant c < \omega$, it is obvious that $\eta_1(b) \leqslant \eta_2(c)$. This completes the proof.

When we put $R_1 = R_2 = R$ and $P_1 = P_2 = P$ in Theorem 2—many comparison theorems for the second-order systems (E_i), $i = 1, 2$, fail to hold for this case—we obtain the following "separation theorem": If R is invertible, $R^{-1} \geqslant 0$, and $P \geqslant 0$ on $[a, \omega)$, then the equation (E) has no nontrivial solution y such that $y(x_1) = y'(x_2) = 0$, $b \leqslant x_1 \leqslant x_2 < \eta(b)$, for any b, $a \leqslant b < \omega$. This result is equivalent to the statement that $\eta(x)$ is a nondecreasing function of x on $[a, \omega)$.

Let $\phi_i(x)$ be the right-hand focal point of x for the equation (E_i), $i = 1, 2$. There are analogous comparison results for $\phi_i(x)$, $i = 1, 2$, which we summarize below.

Let U be the solution of the matrix system

$$\left(R(x)U'\right)' + P(x)U = 0, \qquad U(b) = I, \quad U'(b) = 0,$$

for some b, $a \leqslant b < \omega$. Put $V = -RU'U^{-1}$. If every nontrivial solution y of (E) with $y'(b) = 0$ does not vanish on $[b, \omega)$, then U is invertible on $[b, \omega)$. Thus, V is defined on $[b, \omega)$ and satisfies thereon

$$(\mathrm{MR}') \qquad V' = P + VR^{-1}V, \qquad V(b) = 0.$$

This proves the necessity part of the following theorem.

THEOREM 3. *Suppose that R and P are $n \times n$ matrices with continuous and real elements and that R is invertible on an interval $[a, \omega)$. Let b be a point on $[a, \omega)$. Every nontrivial solution vector y of* (E) *with $y'(b) = 0$ does not vanish on $[b, \omega)$ if and only if the matrix Riccati system* (MR′) *has a solution on $[b, \omega)$.*

Sufficiency of this theorem may be proved in a manner similar to the corresponding proof of Theorem 1, using the following analogue of Lemma 1.

LEMMA 3. *The matrix Riccati equation* (MR′) *has a unique solution on $[b, \phi(b))$, which is continuously differentiable. The solution is nontrivial if $P \not\equiv 0$ and it is nonnegative on $[b, \phi(b))$ if $R^{-1}(x) \geqslant 0$, $P(x) \geqslant 0$, $b \leqslant x < \phi(b)$.*

Using Theorem 3, Lemma 2 (with P_i and R_i^{-1} interchanged in (3), (4) and (5), $i = 1, 2$) and Lemma 3, we can similarly prove the following comparison theorem for $\phi_i(x)$.

THEOREM 4. *If, for some b, $a \leqslant b < \omega$,*

$$P_1(x) \geqslant 0, \qquad \int_b^x P_1(t)\,dt \geqslant \int_b^x P_2(t)\,dt \geqslant 0, \qquad R_1^{-1}(x) \geqslant R_2^{-1}(x) \geqslant 0,$$

$b \leqslant x < \omega$, then $\phi_2(b) \geqslant \phi_1(b)$. Moreover, if

$$P_1(x) \geqslant P_2(x) \geqslant 0, \qquad R_1^{-1}(x) \geqslant R_2^{-1}(x) \geqslant 0,$$

$a \leqslant x < \omega$, then $\phi_2(c) \geqslant \phi_1(b)$, $a \leqslant b \leqslant c < \omega$.

Putting $P_1 = P_2 = P$ and $R_1 = R_2 = R$ in Theorem 4, we again obtain a "separation theorem", which is equivalent to the statement that $\phi(x)$ is a nondecreasing function of x.

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DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK, STONY BROOK, NEW YORK 11794