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A NOTE ON THE STRONG SUMMABILITY OF THE RIESZ MEANS OF MULTIPLE FOURIER SERIES

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ABSTRACT. Let $S_t^\alpha(x,f)$ be the Riesz means of order α of an integrable function $f(x)$ on N-dimensional torus T^N ($N \geqslant 2$), that is,

$$S_t^\alpha(x,f) = \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \hat{f}(m) e^{2\pi i m x}.$$

E. M. Stein has shown that if $1 < p \leqslant 2$ and $\alpha > \alpha_p$ where

$$\alpha_p = \frac{N-1}{2}\left(\frac{2}{p} - 1\right) - \frac{1}{p'} = \frac{N-1}{2} - \frac{N}{p'},$$

then for any function $f(x) \in L^p(T^N)$ $S_t^\alpha(x,f)$ is strong summable to $f(x)$, that is,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |S_t^\alpha(x,f) - f(x)|^2\, dt = 0$$

for almost every x. In this paper we shall show that if $1 \leqslant p \leqslant 2$ and $-1 < \alpha < \alpha_p$, then there exists a function $f(x) \in L^p(T^N)$ such that

$$\frac{1}{T}\int_0^T |S_t^\alpha(x,f)|^2\, dt = \Omega\left(T^{\alpha_p - \alpha}\log^{-2\tau} T\right) \quad \text{as } T \to \infty$$

for every x and every $\tau > 1/p$, in particular,

$$\varlimsup_{T\to\infty} \frac{1}{T}\int_0^T |S_t^\alpha(x,f)|^2\, dt = \infty$$

for every x, where we can take for $f(x)$, $f_{\sigma\tau}(x)$ such that $\hat{f}_{\sigma\tau}(m) = 1/|m|^\sigma \log^\tau |m|$, $|m| > 1$.

1. Introduction. Let R^N denote the N-dimensional Euclidean space, T^N the N-dimensional torus (identified with the cube $Q^N = \{x = (x_1, \ldots, x_N) \in R^N: -\frac{1}{2} \leqslant x_j < \frac{1}{2},\ j = 1, \ldots, N\}$) and Z^N the integral lattice of R^N. Throughout this paper assume $N \geqslant 2$ and $1 \leqslant p \leqslant 2$.

For any $f(x) \in L^1(T^N)$, define the Riesz means of order α of $f(x)$ by

$$S_t^\alpha(x,f) = \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \hat{f}(m) e^{2\pi i m x}$$

with $\hat{f}(m) = \int_{T^N} f(x) e^{-2\pi i m x}\, dx$, $m \in Z^N$.

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The problem of the strong summability of the Riesz means of multiple Fourier series is one of dealing with the validity of the following:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |S_t^\alpha(x,f) - f(x)|^2\, dt = 0.$$

E. M. Stein [6] has shown that if $f \in L^p(T^N)$ and $\alpha > \alpha_p$ where

$$\alpha_p = \frac{N-1}{2}\left(\frac{2}{p} - 1\right) - \frac{1}{p'} = \frac{N-1}{2} - \frac{N}{p'},$$

then $\lim_{T\to\infty}(1/T)\int_0^T|S_t^\alpha(x,f) - f(x)|^2\, dt = 0$ for almost every x. On the other hand, it seems to us as if the case $\alpha \leqslant \alpha_p$ is unknown except the case $p = 1$ and $\alpha = \alpha_p$. (In [6] Stein has stated the affirmative result without the proof.)

The purpose of this paper is to prove a negative result for the case $\alpha < \alpha_p$. The method consists of joining multiple Fourier series and the *Lattice Point Problem* in analytic number theory by means of a special function $f_{\sigma\tau}(x)$ whose Fourier coefficient is given by $\hat{f}_{\sigma\tau}(m) = 1/|m|^\sigma \log^\tau |m|$. Our main result is the following theorem.

THEOREM. *Suppose* $1 \leqslant p \leqslant 2$ *and* $-1 < \alpha < \alpha_p$. *Then there exists a function* $f(x) \in L^p(T^N)$ *such that*

$$\frac{1}{T}\int_0^T |S_t^\alpha(x,f)|^2\, dt = \Omega\left(T^{\alpha_p - \alpha}\log^{-2\tau} T\right) \quad \text{as } T \to \infty$$

for every x *and every* $\tau > 1/p$, *in particular*,

$$\overline{\lim_{T\to\infty}}\, \frac{1}{T}\int_0^T |S_t^\alpha(x,f)|^2\, dt = \infty$$

for every x, *where we can take* $f_{\sigma\tau}(x)$ *for such a function* $f(x)$.

In the above statement, $g(T) = \Omega(h(T))$ implies $g(T) \neq o(h(T))$.

From this theorem we have directly the following corollary.

COROLLARY. *Suppose* $1 \leqslant p \leqslant 2$ *and* $0 \leqslant \alpha < \alpha_p$. *Then there exists a function* $f(x) \in L^p(T^N)$ *such that*

$$S_t^\alpha(x,f) = \Omega\left(t^{(1/2)(\alpha_p - \alpha)}\log^{-\tau} t\right) \quad \text{as } t \to \infty$$

for every x *and every* $\tau > 1/p$, *in particular*,

$$\overline{\lim_{T\to\infty}}\, |S_t^\alpha(x,f)| = \infty$$

for every x, *where we can take* $f_{\sigma\tau}(x)$ *for such a function* $f(x)$.

REMARK. The existence of $f(x) \in L^p(T^N)$ such that $\overline{\lim}_{t\to\infty}|S_t^\alpha(x,f)| = \infty$ for almost every x and the fact we can take $f_\sigma = f_{\sigma 0}$ for such a function f have been shown for $\alpha = 0$ by Stein and Weiss [7] and for $0 \leqslant \alpha < (N-1)/2$ by Babenko [3] (see also Alimov, Il'in and Nikishin [2]). Using our method, it is easy to show that their exceptional sets are empty (see also Kuratsubo [4]).

2. Lemmas. The proof of the theorem depends on several results from [**1**, **5**]. These are stated here for convenience of the description. The next lemma was proved in [**1**, Lemma 2.2] for the case $s > 0$ and $\tau = 0$.

LEMMA 1. *Suppose $s > -1$ and $s = r + \kappa$ where r is an integer and κ satisfies $0 < \kappa \leqslant 1$. For $\beta > 0$ and a nonnegative number τ, define a function $b(\lambda)$ as $\lambda^\beta \log^\tau \lambda$, 0 according to $\lambda \geqslant e$, $\lambda < e$ respectively. Further, for any numerical series $\sum_{k=1}^\infty a_k$, let σ_λ^s, $\overline{\sigma_\lambda^s}$ be $\sum_{k<\lambda}(1-k/\lambda)^s a_k$, $\sum_{k<\lambda}(1-k/\lambda)^s b(k) a_k$ respectively. Then we have*

$$\overline{\sigma_\lambda^s} = b(\lambda)\sigma_\lambda^s + (-1)^{r+1}\int_0^1 \frac{t^{r+1}}{(r+1)!}\sigma_{t\lambda}^{r+1}\left(\frac{d}{dt}\right)^{r+2}\left[(b(t\lambda)-b(\lambda))(1-t)^s\right] dt$$

and, for some positive constant C,

$$\int_0^1 \frac{t^{r+1}}{(r+1)!}\left|\left(\frac{d}{dt}\right)^{r+2}\left[(b(t\lambda)-b(\lambda))(1-t)^s\right]\right| dt \leqslant Cb(\lambda), \qquad \lambda \geqslant 0.$$

PROOF. The first equality is proved in [**1**, Lemma 2.2]. On the other hand, the second inequality follows from the next.

$$\left|\left(\frac{d}{dt}\right)^{r+2}\left[(b(t\lambda)-b(\lambda))(1-t)^s\right]\right|$$
$$\leqslant Cb(\lambda)\left\{(1-t^\beta)(1-t)^{s-r-2} + \sum_{j=1}^{r+2} t^{\beta-j}(1-t)^{s-r+j-2}\right\}$$

and

$$\int_0^1 t^{r+1}(1-t^\beta)(1-t)^{s-r-2}\,dt < +\infty,$$
$$\int_0^1 t^{r+1+\beta-j}(1-t)^{s-r+j-2}\,dt < +\infty \qquad (1 \leqslant j \leqslant r+2).$$

LEMMA 2. *Under the same notation as Lemma 1, there exists a positive constant C such that*

$$\frac{1}{T}\int_0^T \left|\overline{\sigma_\lambda^s}\right|^2 d\lambda \leqslant Cb(T)^2 \sup_{0<t\leqslant T}\left(\frac{1}{t}\int_0^t |\sigma_\lambda^s|^2\,d\lambda\right), \qquad T > 0.$$

PROOF. From Lemma 1, it follows that

$$\int_0^T \left|\overline{\sigma_\lambda^s}\right|^2 d\lambda \leqslant 2\left\{\int_0^T b(\lambda)^2 |\sigma_\lambda^s|^2\,d\lambda\right.$$
$$\left. + \int_0^T \left|\int_0^1 \frac{t^{r+1}}{(r+1)!}\sigma_{t\lambda}^{r+1}\left(\frac{d}{dt}\right)^{r+2}\left[(b(t\lambda)-b(\lambda))(1-t)^s\right] dt\right|^2 d\lambda\right\}$$
$$= 2\{I_1 + I_2\}.$$

First, by monotone increasing of $b(\lambda)$ we have $I_1 \leqslant b(T)^2 \int_0^T |\sigma_\lambda^s|^2 \, d\lambda$. Next, by Schwarz's inequality, Fubini's theorem, Lemma 1 and its proof we have

$$I_2 \leqslant \int_0^T \left(\int_0^1 \frac{t^{r+1}}{(r+1)!} \left| \left(\frac{d}{dt} \right)^{r+2} \left[(b(t\lambda) - b(\lambda))(1-t)^s \right] \right| dt \right)$$

$$\times \left(\int_0^1 \frac{t^{r+1}}{(r+1)!} |\sigma_{t\lambda}^{r+1}|^2 \left| \left(\frac{d}{dt} \right)^{r+2} \left[(b(t\lambda) - b(\lambda))(1-t)^s \right] \right| dt \right) d\lambda$$

$$\leqslant C \int_0^T b(\lambda)^2 \left(\int_0^1 t^{r+1} \left\{ (1-t^\beta)(1-t)^{s-r-2} + \sum_{j=1}^{r+2} t^{\beta-j} (1-t)^{s-r+j-2} \right\} |\sigma_{t\lambda}^{r+1}|^2 \, dt \right) d\lambda$$

$$\leqslant CTb(T)^2 \int_0^1 t^{r+1} \left\{ (1-t^\beta)(1-t)^{s-r-2} + \sum_{j=1}^{r+2} t^{\beta-j} (1-t)^{s-r+j-2} \right\} \frac{1}{Tt} \int_0^{Tt} |\sigma_\lambda^{r+1}|^2 \, d\lambda \, dt$$

$$\leqslant CTb(T)^2 \sup_{0<t\leqslant T} \left(\frac{1}{t} \int_0^t |\sigma_\lambda^{r+1}|^2 \, d\lambda \right)$$

for a suitable constant C, not necessarily the same at each occurrence. On the other hand, when $r < s < r+1$, let δ be a positive number such that $r + 1 = s + \delta$. Then we have the following well-known relation

$$\sigma_\lambda^{s+\delta} = B(\delta, s+1)^{-1} \int_0^1 (1-t)^{\delta-1} t^s \sigma_{t\lambda}^s \, dt$$

with the beta function B (e.g. [7, p. 269]). By the same calculation with I_2-estimation, we have

$$\frac{1}{t} \int_0^t |\sigma_\lambda^{r+1}|^2 \, d\lambda \leqslant B(\delta, s+1)^{-1} \int_0^1 (1-u)^{\delta-1} u^s \left(\int_0^t |\sigma_{u\lambda}^s|^2 \, d\lambda \right) du$$

$$\leqslant \sup_{0<u\leqslant t} \left(\frac{1}{u} \int_0^u |\sigma_\lambda^s|^2 \, d\lambda \right).$$

These complete the proof of Lemma 2.

The next lemma is a metrical theorem of the *Lattice Point Problem* [**5**, Theorem 1] in the case $\alpha \geqslant 0$, but an examination of the proof shows that it applies without significant change to the present situation.

LEMMA 3. *Suppose $\alpha > -1$. Then we have*

$$\frac{1}{T} \int_0^T \left| \sum_{|m|^2 < t} \left(1 - \frac{|m|^2}{t} \right)^\alpha e^{2\pi i m x} \right|^2 dt = \Omega(T^{(N-1)/2-\alpha})$$

for every x.

Applying Lemma 2 to the case

$$a_k = \sum_{|m|^2 = k} \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|}, \qquad \beta = \frac{\sigma}{2}$$

and $s = \alpha$, we have the following lemma.

LEMMA 4. *Suppose* $\alpha > -1$, $\sigma \geqslant 0$, $\alpha + \sigma < (N-1)/2$ *and* τ *is a nonnegative number. Then we have*

$$\frac{1}{T}\int_0^T \left| \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|} \right|^2 dt = \Omega\left(T^{(N-1)/2-\alpha-\sigma} \log^{-2\tau} T\right)$$

for every x.

3. Proof of theorem. Now let σ be equal to N/p', τ larger than $1/p$ and $f_{\sigma\tau}$ the function such that

$$\hat{f}_{\sigma\tau}(m) = \frac{1}{|m|^\sigma \log^\tau |m|}, \qquad |m| > 1.$$

Then, from Wainger's theorem [**8**, Theorem 7] the function $f_{\sigma\tau}$ belongs to $L^p(T^N)$ and from Lemma 4 we have

$$\frac{1}{T}\int_0^T |S_t^\alpha(x, f_{\sigma\tau})|^2 dt = \frac{1}{T}\int_0^T \left| \sum_{1<|m|^2<t} \left(1 - \frac{|m|^2}{t}\right)^\alpha \frac{e^{2\pi i m x}}{|m|^\sigma \log^\tau |m|} \right|^2 dt$$
$$= \Omega\left(T^{(N-1)/2-\alpha-N/p'} \log^{-2\tau} T\right)$$

for every x. This completes the proof of theorem.

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