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# A STRUCTURE THEOREM FOR THE COMMUTANT OF A CLASS OF CYCLIC SUBNORMAL OPERATORS

MARC RAPHAEL

ABSTRACT. An $m$-measure is defined to be a measure $\mu$ such that the analytic bounded point evaluations of $P^2(\mu)$ is the open unit disk $\mathbf{D}$ in the complex plane, and the weak* closure of the analytic polynomials in $L^\infty(\mu)$ is the set of bounded analytic functions on $\mathbf{D}$. A complete characterization of $P^2(\mu) \cap L^\infty(\mu)$, the commutant of the cyclic subnormal operator of multiplication by $z$ on $P^2(\mu)$, is then obtained.

In this paper a complete characterization is given of the commutant of a class of cyclic subnormal operators closely related to the unilateral shift.

An operator $S$ on a Hilbert space $\mathscr{H}$ is *subnormal* if there is a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator $N$ on $\mathscr{K}$ such that $N(\mathscr{H}) \subseteq \mathscr{H}$ and $S = N|\mathscr{H}$ (the restriction of $N$ to $\mathscr{H}$). The weak* topology on $B(\mathscr{H})$ is the topology which $B(\mathscr{H})$ has as the Banach space dual of the trace class operators [**4**].

A *measure* $\mu$ is always a compactly supported, positive regular Borel measure on the complex plane, $\mathbf{C}$. If $S$ is a cyclic subnormal operator, then there exists a measure $\mu$ such that $S$ is unitarily equivalent to $S_\mu$, the operator of multiplication by $z$ on $P^2(\mu) =$ the closure of the analytic polynomials in $L^2(\mu)$ [**4**]. Yoshino's Theorem [**4**] states that the map from $P^2(\mu) \cap L^\infty(\mu)$ onto $\{S_\mu\}' =$ the commutant of $S_\mu$, given by $\phi \mapsto \phi(S_\mu) =$ multiplication by $\phi$, is an isometric isomorphism and a weak* homeomorphism. For functions $f, g$ in $L^2(\mu)$, $\langle f, g\rangle = \int f\bar{g}\,d\mu$, $\|f\|_2 = (\langle f, f\rangle)^{1/2}$, and $\|f\|_\mu$ denotes the $\mu$-essential supremum of $f$. Let $m$ denote normalized arc length measure on $\partial\mathbf{D}$, the boundary of the open unit disk. Thus $S_m$ is the unilateral shift. A class of measures with many of the properties of $m$ will be defined after some notation is set.

If $\mu$ is a measure, then $B(\mu)$, the set of *bounded point evaluations* of $P^2(\mu)$, consists of those $\lambda$ in $\mathbf{C}$ for which the linear functional $p \mapsto p(\lambda)$ has a bounded extension from the polynomials to $P^2(\mu)$. Equivalently, $\lambda \in B(\mu)$ if and only if there exists $k_\lambda$ in $P^2(\mu)$ such that $p(\lambda) = \langle p, k_\lambda\rangle$ for all polynomials $p$. $B_a(\mu)$, the set of *analytic bounded point evaluations* of $P^2(\mu)$, is the largest open subset of $B(\mu)$ such that the function

$$\tilde{f}(\lambda) = \langle f, k_\lambda\rangle \tag{1}$$

is analytic in $B_a(\mu)$ for every $f$ in $P^2(\mu)$.

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In view of (1), every $f$ in $P^2(\mu)$ can be given a pointwise definition on $B_a(\mu)$; namely, $f(\lambda) = \tilde{f}(\lambda)$ for $\lambda$ in $B_a(\mu)$. Since the pointwise definition of $f$ agrees $\mu$-a.e. with any Borel function that represents the equivalence class of $f$ in $P^2(\mu)$, it will be agreed upon once and for all that each $f$ in $P^2(\mu)$ is defined pointwise on $B_a(\mu)$ via (1). The $\sim$ notation will be dropped in all instances.

The proof of the following proposition is left to the reader.

PROPOSITION 2. *Let $\mu$ be a measure and $\lambda \in B_a(\mu)$. If $f \in P^2(\mu)$ and $g \in P^2(\mu) \cap L^\infty(\mu)$, then $fg \in P^2(\mu)$ and $(fg)(\lambda) = f(\lambda)g(\lambda)$.*

An *m-measure* is defined as a measure $\mu$ with the following two properties:

(a) $B_a(\mu) = \mathbf{D}$;

(b) $P^\infty(\mu)$, the weak* closure of the analytic polynomials in $L^\infty(\mu)$, has no $L^\infty$-summand, and the interior of the Sarason hull of $\mu$ is $\mathbf{D}$.

The terminology of condition (b) is taken from [4]. Condition (b) is equivalent to the identity mapping on the polynomials extending to an isometric isomorphism that is a weak* homeomorphism from $P^\infty(\mu)$ onto $P^\infty(m)$. In particular $\mu$ is supported on $\mathbf{D}$ and $\mu|\partial\mathbf{D} << m$.

Condition (b) will be abbreviated simply to "$P^\infty(\mu) = H^\infty$". It is not a substantial restriction on $\mu$ as is indicated by Theorem 4.11 of [5]. On the other hand, if $P^2(\mu) \neq L^2(\mu)$, then it is unknown whether $B(\mu)$ is nonempty.

EXAMPLE 3. Some examples of $m$-measures are now given.

(i) Of course $m$ is an $m$-measure.

(ii) Area measure on $\mathbf{D}$ is an $m$-measure.

(iii) If $\mu$ is a measure supported on $\mathbf{D}^-$ such that $\mu|\partial\mathbf{D} << m$ and $d(\mu|\partial\mathbf{D})/dm$ is log-integrable with respect to $m$, then $\mu$ is an $m$-measure. This follows by combining Theorem 4.5 of [2] with Corollary 5 of [8].

(iv) Let $\{a_n\}$ be a sequence in $\mathbf{D}$ with $\partial\mathbf{D} \subseteq \{a_n\}^-$. If almost every point of $\partial\mathbf{D}$ can be approached nontangentially by a subsequence of $\{a_n\}$, then $\mu = \Sigma 2^{-n}\delta_{a_n}$ is an $m$-measure [1]. Here, $\delta_{a_n}$ is the unit point mass measure at $a_n$.

(v) If $\nu$ is a probability measure on $[0, 1]$ with 1 in the support of $\nu$, then $d\mu(re^{i\theta}) = d\nu(r)\, dm(e^{i\theta})$ is an $m$-measure and $S_\mu$ is the canonical model for a subnormal weighted shift operator of norm 1 [4].

(vi) Let $G = \mathbf{D} \setminus \{z \in \mathbf{C}\colon |z - 1/2| \leqslant 1/2\}$. Using a technique from [6], one can construct a measure $\mu$, equivalent to area measure on $G$, such that $P^\infty(\mu) = H^\infty$ and $B_a(\mu) = G$. Thus $\mu$ is not an $m$-measure since $B_a(\mu) \neq \mathbf{D}$.

The main result of this paper is the following.

THEOREM 4. *If $\mu$ is an m-measure, then there is a unique $\mu$-measurable subset $\Delta$ of $\partial\mathbf{D}$ such that*:

(a) $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu|\mathbf{C} \setminus \Delta) \oplus L^\infty(\mu|\Delta)$;

(b) $\mu|\mathbf{C} \setminus \Delta$ *is an m-measure*;

(c) $P^2(\mu|\mathbf{C} \setminus \Delta) \cap L^\infty(\mu|\mathbf{C} \setminus \Delta) = P^\infty(\mu|\mathbf{C} \setminus \Delta) = H^\infty$.

Theorem 4 will be established with the aid of several lemmas.

LEMMA 5. *If $\mu$ is an m-measure and $f \in P^2(\mu) \cap L^\infty(\mu)$, then there is a $g$ in $P^\infty(\mu)$ such that $(f-g)|\mathbf{D} = 0$.*

PROOF. It can be assumed that $\|f\|_\mu < 1$. For $\lambda$ in $\mathbf{D}$,

$$|f(\lambda)|^n = |\langle f, k_\lambda\rangle|^n = |\langle f^n, k_\lambda\rangle| \leqslant \|f^n\|_2 \|k_\lambda\|_2 \leqslant \|k_\lambda\|_2.$$

Letting $n \to \infty$ shows that $|f(\lambda)| \leqslant 1$. Since $B_a(\mu) = \mathbf{D}$, $f$ is analytic and bounded on $\mathbf{D}$. Since $P^\infty(\mu) = H^\infty$, there exists $g$ in $P^\infty(\mu)$ such that $(f-g)|\mathbf{D} = 0$. □

LEMMA 6. *If $\mu$ is a measure such that $S_\mu$ is quasisimilar to $S_m$, then $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu) = H^\infty$.*

PROOF. According to Theorem 4.5 of [2], a necessary and sufficient condition for $S_\mu$ and $S_m$ to be quasisimilar is that $\mu$ be a measure as in Example 3(iii). Therefore $P^\infty(\mu) = H^\infty$ and if $f \in P^2(\mu) \cap L^\infty(\mu)$, then there exists, by the previous lemma, a $g$ in $P^\infty(\mu)$ such that $(f-g)|\mathbf{D} = 0$. Let $h = f - g$ and choose a sequence of polynomials $\{p_n\}$ that converges to $h$ in $L^2(\mu)$. Choose $\phi$ in $H^2$ (the Hardy space of square integrable functions) such that $|\phi|^2 = d\mu_0/dm$ where $\mu_0 = \mu|\partial\mathbf{D}$. Then

$$0 = \lim_{n\to\infty} \int_{\partial\mathbf{D}} |p_n - h|^2 \, d\mu = \lim_{n\to\infty} \int |p_n\phi - h\phi|^2 \, dm.$$

It follows that $h\phi \in H^2$; so if $P_\lambda$ is the Poisson kernel at $\lambda$, then

$$(h\phi)(\lambda) = \lim_{n\to\infty} \int P_\lambda(p_n\phi)\, dm = \lim_{n\to\infty} (p_n\phi)(\lambda) = 0.$$

Thus $h\phi = 0$ as an element of $H^2$. Since $\log|\phi| \in L^1(m)$, it must be that $h = 0$ $m$-a.e.; hence $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu)$. □

LEMMA 7. *Let $\mu$ be an m-measure and $g \in P^2(\mu) \cap L^\infty(\mu)$. If $g|\mathbf{D} = 0$ and $E = \{z \in \partial\mathbf{D}\colon g(z) \neq 0\}$, then $\chi_E \in P^2(\mu) \cap L^\infty(\mu)$.*

PROOF. If $S_\mu$ is quasisimilar to $S_m$, then Lemma 6 implies that $P^2(\mu) \cap L^\infty(\mu) = H^\infty$. In this case, $\chi_E = 0$, so there is nothing to prove.

Now suppose $S_\mu$ is not quasisimilar to $S_m$. Then $\log(d\mu_0/dm) \notin L^1(m)$ where $\mu_0 = \mu|\partial\mathbf{D}$ (Theorem 4.5, [2]). Hence Szegö's Theorem implies that $P^2(\mu_0) = L^2(\mu_0)$. For $k > 0$ let $E_k = \{z \in \partial\mathbf{D}\colon |g(z)| > k^{-1}\}$, and let $\{p_n\}$ be a sequence of polynomials such that

$$0 = \lim_{n\to\infty} \int |p_n - g^{-1}\chi_{E_k}|^2 \, d\mu_0 = \lim_{n\to\infty} \int |p_n g - \chi_{E_k}|^2 \, d\mu.$$

Thus $\chi_{E_k} \in P^2(\mu)$. Since $\chi_{E_k} \to \chi_E$ weak* in $L^\infty(\mu)$, $\chi_E \in P^2(\mu) \cap L^\infty(\mu)$. □

PROOF OF THEOREM 4. If $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu)$, then it is easy to see that the theorem holds, and $\Delta$ is the empty set.

Now suppose $P^2(\mu) \cap L^\infty(\mu) \neq P^\infty(\mu)$. By Lemma 6 and Theorem 4.5 of [2], $\log(d\mu_0/dm) \notin L^1(m)$ where $\mu_0 = \mu|\partial\mathbf{D}$. Szegö's Theorem now shows that

$$\mu(\partial\mathbf{D}) < \mu(\mathbf{D}^-). \tag{8}$$

A standard argument shows that $W = \{F \subseteq \partial\mathbf{D}\colon \chi_F \in P^2(\mu) \cap L^\infty(\mu)\}$ contains a unique maximal element, $\Delta$. Since $P^2(\mu) \cap L^\infty(\mu) \neq P^\infty(\mu)$, Lemmas 5 and 7 imply $\mu(\Delta) > 0$.

It is now shown that $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu|\mathbf{C} \setminus \Delta) \oplus L^\infty(\mu|\Delta)$. If $f \in L^\infty(\mu|\Delta)$ let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $L^2(\mu_0) = P^2(\mu_0)$. Since $p_n\chi_\Delta \to f$ in $L^2(\mu)$, $f \in P^2(\mu) \cap L^\infty(\mu)$. Now suppose that $f \in P^\infty(\mu|\mathbf{C} \setminus \Delta)$. Let $\{p_\alpha\}$ be a net of polynomials that converges to $f$ weak* in $L^\infty(\mu|\mathbf{C} \setminus \Delta)$. Since the net $\{p_\alpha\}$ converges to $f$ weak* in $L^2(\mu|\mathbf{C} \setminus \Delta)$, a sequence consisting of convex combinations of elements from the set $\{(1 - \chi_\Delta)p_\alpha\}$ converges to $f$ in $L^2(\mu)$. This shows that $P^\infty(\mu|\mathbf{C} \setminus \Delta) \oplus L^\infty(\mu|\Delta) \subseteq P^2(\mu) \cap L^\infty(\mu)$.

To show the reverse inclusion, let $f \in P^2(\mu) \cap L^\infty(\mu)$ and select $g$ in $P^\infty(\mu)$ such that $(f - g)|\mathbf{D} = 0$ (Lemma 5). If $E = \{z \in \partial\mathbf{D}\colon (f - g)(z) \neq 0\}$, then according to Lemma 7,

$$\chi_E \in P^2(\mu) \cap L^\infty(\mu).$$

By the maximality of $\Delta$, $E \subseteq \Delta$; so $(1 - \chi_\Delta)(f - g) = 0$. Therefore $(1 - \chi_\Delta)f = (1 - \chi_\Delta)g \in P^\infty(\mu|\mathbf{C} \setminus \Delta)$. Since $f = (1 - \chi_\Delta)f + \chi_\Delta f$, (a) of Theorem 4 holds.

Now, for notational convenience, let $\nu = \mu|\mathbf{C} \setminus \Delta$, $\nu' = \mu|\Delta$, and $\sigma_{\mathrm{ap}}(S)$ be the approximate point spectrum of $S$. Since $S_\mu = S_\nu \oplus S_{\nu'}$ and $S_{\nu'}$ is normal, it follows that

$$\sigma(S_\nu) \setminus \sigma_{\mathrm{ap}}(S_\nu) \supseteq \sigma(S_\nu) \setminus \sigma_{\mathrm{ap}}(S_\mu) = \sigma(S_\mu) \setminus \sigma_{\mathrm{ap}}(S_\mu).$$

By Theorem 1.1 of **[9]**, $B_a(\nu) \supseteq B_a(\mu) = \mathbf{D}$; hence $B_a(\nu) = \mathbf{D}$. Since, in general, $B_a(\nu)$ is contained in the interior of the Sarason hull of $\nu$, it will follow that $\nu$ is an $m$-measure if $P^\infty(\nu)$ has no $L^\infty$-summand. Suppose $L^\infty(\nu|\Sigma)$ is an $L^\infty$-summand of $P^\infty(\nu)$ (viz. $P^\infty(\nu) = P^\infty(\nu|\mathbf{C} \setminus \Sigma) \oplus L^\infty(\nu|\Sigma)$). Since $B_a(\nu) = \mathbf{D}$, $\Sigma \subseteq \partial\mathbf{D} \setminus \Delta$. This contradicts the maximality of $\Delta$; thus (b) of Theorem 4 holds.

To obtain (c), apply what has already been proved to the $m$-measure $\mu|\mathbf{C} \setminus \Delta$ and use the maximality of $\Delta$.

To prove uniqueness of $\Delta$ suppose that $\Delta_1$ is any subset of $\partial\mathbf{D}$ that satisfies the conditions of the theorem. Since $\chi_{\Delta_1} \in P^2(\mu) \cap L^\infty(\mu)$ and $\Delta$ is the unique maximal element of $W$, $\Delta_1 \subseteq \Delta$. By (c), $\chi_{\Delta \setminus \Delta_1} \in P^2(\mu|\mathbf{C} \setminus \Delta_1) \oplus L^\infty(\mu|\Delta_1) = H^\infty$. If $\chi_{\Delta \setminus \Delta_1} \neq 0$, then $\mu(\Delta \setminus \Delta_1) = \mu(\partial\mathbf{D})$. It then follows that $\chi_{\Delta \setminus \Delta_1}(z) = 1$ for every $z$ in $\mathbf{D}$. This contradicts (8) and establishes the theorem. □

This paper concludes with four easy consequences of Theorem 4. The following notation will be useful.

NOTATION 9. (i) Assume $\mu$ is an $m$-measure and $\phi \in P^2(\mu) \cap L^\infty(\mu)$.

(ii) $N_\mu$ is the normal operator of multiplication by $z$ on $L^2(\mu)$.

(iii) $\phi(S_\mu)$ and $\phi(N_\mu)$ are the operators of multiplication by $\phi$ on $P^2(\mu)$ and $L^2(\mu)$, respectively.

(iv) If $\mu$ is an $m$-measure and $\Delta$ is as in Theorem 4, then $\mu_1 = \mu|\mathbf{C} \setminus \Delta$ and $\mu_2 = \mu|\Delta$. The decomposition of $\phi$ with respect to $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu_1) \oplus L^\infty(\mu_2)$ is $\phi = \phi_1 \oplus \phi_2$.

COROLLARY 10. *Suppose that $\mu$ and $\nu$ are $m$-measures. If $S_\mu$ and $S_\nu$ are quasisimilar, then $\{S_\mu\}'$ is isometrically isomorphic and weak* homeomorphic to $\{S_\mu\}'$ via a map that takes $S_\mu$ to $S_\nu$.*

PROOF. Let $\Delta$, $\Sigma \subseteq \partial\mathbf{D}$ be the sets given by Theorem 4 in the decomposition of $P^2(\mu) \cap L^\infty(\mu)$ and $P^2(\nu) \cap L^\infty(\nu)$, respectively. The decomposition of $S_\mu$ and $S_\nu$ into their pure and normal parts is

$$S_\mu = S_{\mu_1} \oplus N_{\mu_2} \quad \text{and} \quad S_\nu = S_{\nu_1} \oplus N_{\nu_2}.$$

Since $N_{\mu_2}$ and $N_{\nu_2}$ are unitarily equivalent [3], $L^\infty(\mu_2) = L^\infty(\nu_2)$. The map is now obvious since $P^\infty(\mu_1) = H^\infty = P^\infty(\nu_1)$. $\square$

COROLLARY 11. *Let $\mu$ be an m-measure and $\phi = \phi_1 \oplus \phi_2$ be in $P^2(\mu) \cap L^\infty(\mu)$. Then*:

(a) *the minimal normal extension of $\phi(S_\mu)$ is $\phi(N_\mu)$ if and only if $\phi_1$ is not constant*;
(b) *if $\phi_1$ is constant, then $\phi(S_\mu)$ is normal*;
(c) $\sigma(\phi(S_\mu)) = \phi(\mathbf{D})^- \cup \{\mu_2$*-essential range of* $\phi_2\}$.

PROOF. Corollary 11 is a direct result of Corollary 3.2 and Theorem 4.2 of Chapter VIII of [4]. $\square$

The proof of the next corollary is left to the reader.

COROLLARY 12. *Let $\mu$ be an m-measure. A subspace $\mathcal{M}$ of $P^2(\mu)$ is a hyperinvariant subspace for $S_\mu$ if and only if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_1$ is an invariant subspace for $S_{\mu_1}$ and $M_2$ is a reducing subspace for $N_{\mu_2}$.*

If $\mu$ is any measure, Theorems 2 and 3 of [7] show that $S_\mu$ has an invariant subspace that is not hyperinvariant if and only if $P^2(\mu) \cap L^\infty(\mu) \neq P^\infty(\mu)$. Thus Theorem 4 yields the following result.

COROLLARY 13. *Let $\mu$ be an m-measure and $\Delta$ be as in Theorem 4. Then $S_\mu$ has an invariant subspace that is not hyperinvariant if and only if $\Delta$ is nonempty.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, ROLLA, MISSOURI 65401