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ON LINKING COEFFICIENTS

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ABSTRACT. The possible values of linking coefficients for two component links are studied. An example of a link $S^3 \cup S^2$ in S^5 having linking coefficient in $\pi_3(S^2)$ of Hopf invariant two is constructed. A generalization to links $S^{m-2} \cup S^p$ in S^m is obtained in the metastable range. Applications to embeddings of two cell complexes are cited.

0. Introduction and statement of results. Let $f\colon S^a \to S^m$, $g\colon S^b \to S^m$ be a pair of disjoint (pl) embeddings of spheres (a two component link). The complement $S^m \setminus g(S^b)$ has the homotopy type of S^{m-b-1} if either $b \leqslant m-3$ or $b = m-2$ and g is unknotted. (If $b \leqslant m-3$, then by Zeeman [**Z**] g is unknotted.) In either case, f represents an element of $\pi_a(S^{m-b-1})$ called a linking coefficient, which we denote by $L(f, g)$.

Problem A. What values can $L(f, g)$ take?

The following theorem restricts the possible values of $L(f, g)$ (cf. Kervaire [**K**]).

THEOREM 1. *Suppose $L(f,g)$ and $L(g,f)$ are both defined. Then $\Sigma^b L(f,g) = \pm\Sigma^a L(g,f)$ in $\pi_{a+b}(S^{m-1})$, where Σ^b, Σ^a are iterated suspension homomorphisms.*

Note that $\pi_{a+b}(S^{m-1})$ is a stable homotopy group. Hence,

COROLLARY 2. *Suppose $m-a \geqslant 3$, $m-b \geqslant 3$. A necessary condition for $L \in \pi_a(S^{m-b-1})$ to be a linking coefficient is that L stably desuspends to $\pi_b(S^{m-a-1})$.*

Using Haefliger's link classification theorem [**H**] in codimension greater than two, the above condition is seen to be sufficient in a range.

THEOREM 3. *Suppose $m-a \geqslant 3$, $m-b \geqslant 3$ and $2a+2b \leqslant 3m-6$. Suppose L stably desuspends to $\pi_b(S^{m-a-1})$. Then L is a linking coefficient for some link $S^a \cup S^b$ in S^m.*

W. Massey has asked: What can be said if one of the factors is allowed to have codimension 2? For links $f\colon S^{m-2} \to S^m$, $g\colon S^p \to S^m$, $q = m-q-1 \geqslant 2$, we have $L(f,g)$ defined in $\pi_{m-2}(S^q)$. However, because of possible knotting of the

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codimension 2 factor, $L(g, f)$ need not be defined. Nevertheless, Levine (unpublished; cf. [**F**] for an alternative proof) has shown that Kervaire's proof can be modified to show that $L(f, g)$ is stably trivial ($p \geqslant 2$). A stronger result is in fact true:

THEOREM 4. *Let $f\colon S^{m-2} \to S^m$, $g\colon S^p \to S^m$ be disjoint embeddings $q = m - p - 1 \geqslant 2$, $p \geqslant 2$, Then the (single) suspension $\Sigma L(f, g) \in \pi_{m-1}(S^{q+1})$ of $L(f, g)$ is zero.*

This necessary condition is sufficient in the metastable range:

THEOREM 5. *Suppose $p \geqslant 2$, $q = m - p - 1 \geqslant 2$ and $m - 2 \leqslant 3q - 3$. Suppose $L \in \pi_{m-2}(S^q)$ satisfies $\Sigma L = 0$, where $\Sigma L \in \pi_{m-1}(S^{q+1})$ is the suspension of L. Then there is a link $S^{m-2} \cup S^p$ with linking coefficient L. Moreover, S^{m-2} may be taken to be unknotted.*

REMARK. Since S^{m-2} may be taken unknotted, it follows that S^{m-2} can have any knot type. This is surprising since it had been thought that knotting phenomena might give rise to "obstructions".

Links $S^3 \cup S^2$ are a particular case of Theorem 5. Theorems 4 and 5 can be restated as follows:

COROLLARY 6. *$\alpha \in \pi_3(S^2)$ is the linking coefficient of some $S^3 \cup S^2$ in S^5 if and only if α has even Hopf invariant.*

There is a close relationship between links and embeddings of two cell complexes up to homotopy type. Let $S^{n-1} \to S^m \setminus S^p$ be an embedding. Consider $S^m \subset D^{m+1} \subset \mathbf{R}^{m+1}$. Then "coning off" the embedding yields an embedding up to homotopy type of $S^q \cup_\alpha D^n$ in $\mathbf{R}^{m+1}$, where α is the homotopy class of $S^{n-1} \to S^m \setminus S^p \simeq S^q$, $q = m - p - 1 \geqslant 2$. Therefore, Theorem 5 yields

COROLLARY 7. *Suppose $q \geqslant 2$, $n - q \geqslant 2$ and $n \leqslant 3q - 2$. Let $\alpha \in \pi_{n-1}(S^q)$ satisfy $\Sigma\alpha = 0$. Then $S^q \cup_\alpha e^n$ embeds up to homotopy type in $\mathbf{R}^{n+2}$.*

It seems likely that the condition $\Sigma\alpha = 0$ is necessary. We can prove

THEOREM 8. *If $S^q \cup_\alpha e^n$ embeds in $\mathbf{R}^{n+2}$, $n - q \geqslant 2$, then α is stably trivial.*

COROLLARY 9. *$S^2 \cup_\alpha e^4$ embeds in $\mathbf{R}^6$ up to homotopy type if and only if α has even Hopf invariant.*

(That $S^2 \cup_\alpha e^4$ does not embed in $\mathbf{R}^6$, for α of odd Hopf invariant, is classical and due to Thom [**T**].)

This paper is organized as follows. In §1 we recall known results in codimension greater than 2. In §2 we given an explicit embedding of $S^3 \cup S^2$ in S^5 with linking coefficient of Hopf invariant 2. The more general result, Theorem 5, is proved in §3. We make use of explicit embeddings of Hacon [**Hac**] and calculate their linking coefficients. Finally, proofs of Theorems 4 and 8 are given in §4. They both have the same homotopy theoretical flavor.

The interested reader may refer to [**Ke**] for a generalization of Hacon's results.

1. Haefliger's classification sequence. The following result is due to Zeeman (cf. **[Z]**).

PROPOSITION. *Suppose $a \leqslant b$. Then any $L \in \pi_a(S^{m-b-1})$ is the linking coefficient of some link.*

PROOF. If $g\colon S^b \to S^m$ is the standard embedding, then $S^m \setminus g(S^b) = S^{m-b-1} \times \mathbf{R}^{b+1}$. Let $\varphi\colon S^q \to S^{m-b-1}$ represent L, and let $\psi\colon S^a \to \mathbf{R}^{b+1}$ be the standard embedding $S^a \subset \mathbf{R}^{a+1} \subset \mathbf{R}^{b+1}$ (or any embedding). Then $f = (\varphi, \psi)\colon S^a \to S^{m-b-1} \times \mathbf{R}^{b+1}$ is also an embedding, and $L(f, g) = L$.

In **[H]**, Haefliger reduces the classification problem for links of codimension greater than 2 to a problem in homotopy theory. As a particular case, he proves

THEOREM (HAEFLIGER **[H]**). *Suppose $m - a \geqslant 3$, $m - b \geqslant 3$, $b \leqslant a$ and $b + 3a \leqslant 3m - 6$. Let $L^m_{a,b}$ denote the group of isotopy classes of pl links $S^a \cup S^b$ in S^m. There is an exact sequence*

$$L^m_{a,b} \xrightarrow{\lambda} \pi_a(S^{m-b-1}) \oplus \pi_b(S^{m-a-1}) \xrightarrow{\omega} \pi_{m-2}(S^{2m-a-b-3}),$$

where $\lambda(f, g) = (L(f, g), L(g, f))$ and ω is the iterated suspension (up to sign) on each factor.

REMARKS. (1) W. Massey has pointed out that the hypothesis $b + 3a \leqslant 3m - 6$ may be replaced by the weaker hypothesis $2b + 2a \leqslant 3m - 6$. This is just the hypothesis necessary so that the group $\pi_{m-2}(S^{2m-a-b-3})$ is stable. The proof is as in **[H]**.

(2) Theorem 3 is an immediate consequence of the exactness of this sequence and the above remark.

(3) The homomorphism ω expresses a symmetry in the linking coefficients, which we recall here for use later.

Let M be the boundary connected sum of $S^a \times D^{m-a}$ and $S^b \times D^{m-b}$. Given an embedding of $S^a \cup S^b$ in S^m, by Zeeman (since each component is unknotted) we may extend the embeddings to framed embeddings and hence to an embedding of M (unique up to concordance).

Let $a' = m - a - 1$, $b' = m - b - 1$. Then $\partial M = S^a \times S^{a'} \# S^b \times S^{b'}$. One can easily check that the inclusion $S^{a'} \vee S^{b'} \to \overline{S^m \setminus M}$ is a homotopy equivalence. Restricting a homotopy inverse to ∂M yields a retraction $\lambda\colon S^a \times S^{a'} \# S^b \times S^{b'} \to S^{a'} \vee S^{b'}$.

Let λ_a, λ_b be the homotopy classes of the maps $\lambda|_{S^a}$, $\lambda|_{S^b}$. These are generalized linking invariants in $\pi_*(S^{a'} \vee S^{b'})$. Let i_a denote the homotopy class of the inclusion $S^{a'} \to S^{a'} \vee S^{b'}$, and let p_a denote the homotopy class of the projection $S^{a'} \vee S^{b'} \to S^{a'}$; similarly for i_b, p_b. Then λ_a, λ_b satisfy

(1) $p_a \circ \lambda_a = 0$, $p_b \circ \lambda_b = 0$,

(2) $p_b \circ \lambda_a = L(f, g)$, $p_a \circ \lambda_b = L(g, f)$,

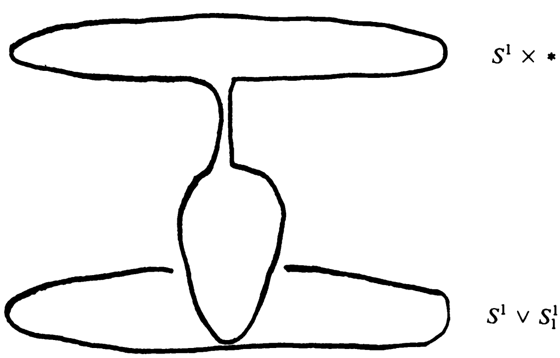
(3) $[\lambda_a, i_a] + [\lambda_b, i_b] = 0$, where $[\ ,\ ]$ denotes Whitehead product.

Condition (3) is the desired symmetry relation.

2. An example of a link of S^3 and S^2 in S^5. Consider $S^2 \times \mathbf{R}^3 = S^5 \setminus S^2 \subset S^5$. Let f: $S^2 \vee S^2 \to S^2 \times \mathbf{R}^3$ be any embedding which up to homotopy is just the folding map Λ: $S^2 \vee S^2 \to S^2$. Suppose f extends in S^5 to an embedding $S^2 \times S^2 \to S^5$. Some small neighborhood X of $S^2 \vee S^2$ in $S^2 \times S^2$ will be embedded into $S^2 \times \mathbf{R}^3$. The resulting embedding $S^3 = \partial X \to X \to S^2 \times R^3 = S^5 \setminus S^2$ will have Hopf invariant two (since $\partial X \to X$ is, up to homotopy, the Whitehead product $[i_1, i_2]$: $S^3 \to S^2 \vee S^2$ and $\Lambda \circ [i_1, i_2] = [i, i]$ has Hopf invariant two). Moreover, since ∂X bounds a disk in $S^2 \times S^2$, this embedding is unknotted in S^5.

It is an easy matter to embed $S^2 \vee S^2$ as above: Let $D_1^3 \subset \mathbf{R}^3$ be a disk and $S_1^2 = \partial D_1^3$. The embedding $S^2 \vee S_1^2 \subset S^2 \times S_1^2 \subset S^2 \times \mathbf{R}^3$ satisfies all the conditions above except that $S_1^2 \to S^2 \times \mathbf{R}^3$ is of degree zero. Choose $* \in \mathbf{R}^3$ outside of D_1^3. Define a new embedding $S^2 \vee S_1^2 \# S^2 \times *$ (see the diagram), where # denotes the connected sum along some path joining the two spheres. This now has degree 1 on the second factor. Moreover, since $S^2 \times *$ bounds a disk in $S^5 \setminus S^2 \times D_1^3$, the two embeddings are ambient isotopic in S^5. In particular, the new embedding also extends to $S^2 \times S^2$, as required.

An embedding of $S^1 \vee S^1$ in $S^1 \times \mathbf{R}^2 \subset \mathbf{R}^3$



REMARK. The link $S^3 \cup S^2 \subset S^5$ may also be obtained as follows: Note that since $f(S^3) \subset S^5$ is unknotted, $S^5 \setminus f(S^3) = S^1 \times \mathbf{R}^4$. Then g: $S^2 \to S^1 \times \mathbf{R}^4$ is an embedding which, while homotopically trivial, cannot be isotopic to the trivial embedding.

In **[Hac]**, Hacon shows that embeddings $S^2 \hookrightarrow S^1 \times \mathbf{R}^4$ are classified up to isotopy by the following: Let $\widetilde{S^1 \times \mathbf{R}^4} = \mathbf{R}^5$ be the universal cover with transformation group $\langle T \rangle = \mathbf{Z}$. Then if $\tilde{g}$: $S^2 \to S^1 \times \mathbf{R}^4$ is any lift of g: $S^2 \to S^1 \times \mathbf{R}^4$, the linking numbers $L(\tilde{g}, T^k\tilde{g}) \in \mathbf{Z}$, $k \neq 0$, are defined and nonzero for only finitely many k. The map $\mathrm{Emb}(S^2, S^1 \times \mathbf{R}^4) \to \oplus_{k=1}^{\infty} \mathbf{Z}$ is a bijection.

Follow the isotopy in the construction of the link backwards. One gets an isotopy of $g(S^2)$ (where $S^5 \setminus g(S^2) = S^2 \times \mathbf{R}^3$) to a new embedding g': $S^2 \to S^5$ in the complement of $S^2 \vee S_1^2$ linking each wedge factor once. Up to isotopy (since any two such are isotopic) it is precisely the connected sum of $g(S^2)$ and $S^2 \times 0 \subset S^2 \times D^3 \subset S^2 \times \mathbf{R}^3$. Note that $g(S^2)$ and $S^2 \times 0$ link once. Since the path connecting them travels once through $D^3 = S^2 \times S^2 \setminus S^2 \vee S^2$, we get

$$L(\tilde{g}', T^k\tilde{g}') = \begin{cases} 1, & k = 1, \\ 0, & k > 1. \end{cases}$$

3. Proof of Theorem 5. By Hacon [**Hac**], embeddings of S^p in $S^1 \times R^{m-1} = S^m \setminus S^{m-2}$ form a group isomorphic to $\oplus_{k=1}^{\infty} \pi_p(S^q)$, $q = m - p - 1$, valid in the metastable range $3(p+1) < 2m$; i.e., $m - 2 \leqslant 3q - 3$. If $\langle T \rangle = \pi_1(S^1 \times \mathbf{R}^{m-1})$ denotes the deck transformation group for the universal covering $\mathbf{R}^m = S^1 \times \mathbf{R}^{m-1}$, the Hacon invariants are the linking coefficients $L(\tilde{g}, T^k\tilde{g}) \in \pi_p(S^q)$, where $\tilde{g}$: $S^p \to R^m = S^1 \times R^{m-1}$ denotes any lift of g: $S^p \to S^1 \times \mathbf{R}^{m-1}$. Note that this is independent of the choice of lifts.

Associate to any embedding g: $S^p \to S^1 \times \mathbf{R}^{m-1} = S^m \setminus S^{m-2}$ a link (f, g): $S^{m-2} \cup S^p \subset S^m$. One obtains from this link the linking coefficient $L(f, g)$ in $\pi_{m-2}(S^{m-p-1})$.

Using Hacon's isomorphism, this gives a map $\oplus_{k=1}^{\infty} \pi_p(S^q) \to \pi_{m-2}(S^q)$ which can be calculated:

PROPOSITION. *The above map is given by the formula*

$$\bigoplus \beta_k \to [i, i] \circ \sum_{k=1}^{\infty} k\tilde{\beta}_k,$$

where $\tilde{\beta}_k \in \pi_{m-2}(S^{2q-1})$ *is (up to sign) the iterated suspension of* β_k.

To prove Theorem 5 from the Proposition, note that under the hypothesis of the theorem, $\pi_p(S^q)$ is a stable group. Taking $\beta_1 = \beta$ and $\beta_k = 0$, $k \neq 1$, we see that the right side is of the form $[i, i] \circ \tilde{\beta}$, with $\tilde{\beta}$ any arbitrary element of $\pi_{m-2}(S^{2m-2p-3})$. Furthermore, under the hypothesis $m - 2 \leqslant 3q - 3$, the EHP sequence [**W**, pp. 548–549] is valid. It follows that any element in the kernel of the suspension homomorphism is of the form $[i, i] \circ \tilde{\beta}$. Thus if L is in the kernel, let $L = [i, i] \circ \tilde{\beta}$. Then L is the linking coefficient of the link $S^{m-2} \cup S^p$ having Hacon invariant $\beta_1 = \beta$, $\beta_k = 0$, $k > 1$.

PROOF OF THE PROPOSITION. As in §1, Remark (3), extend the embedding $S^{m-2} \cup S^p \subset S^m$ to an embedding of the compact manifold M, boundary connected sum of $S^{m-2} \times D^2$ and $S^p \times D^{q+1}$. $\partial M = S^{m-2} \times S^1 \# S^p \times S^q$, and the inclusion $S^1 \vee S^q \to \overline{S^m \setminus M}$ is again a homotopy equivalence. The restriction of a homotopy inverse to ∂M yields a retraction λ: $S^{m-2} \times S^1 \# S^p \times S^q \to S^1 \vee S^q$.

The Hacon invariants are contained in the homotopy class of the map (denoted by β) $\lambda|_{S^p}$: $S^p \to S^1 \vee S^q$. The desired linking coefficient $L(f, g)$ is obtained from the homotopy class (denoted by α) of the map $\lambda|_{S^{m-2}}$: $S^{m-2} \to S^1 \vee S^q$ by composing with the projection to S^q. Furthermore, α and β are related by the formula

$$(T - 1)\alpha + [\beta, i] = 0,$$

where i is the homotopy class of the inclusion $S^q \to S^1 \vee S^q$, and $\langle T \rangle = \pi_1(S^1 \vee S^q)$ acts on $\pi_*(S^1 \vee S^q)$ in the usual fashion. Now since $T - 1$ is injective, we are able to calculate α from β.

Under the dimension restrictions, $m-2 \leqslant 3q-3$ (i.e., $p < 2q-1$), $\pi_p(S^1 \vee S^q) = \oplus_{k=-\infty}^{\infty} \pi_p(S^q)$, and so we may write $\beta = \Sigma_{k=-\infty}^{\infty} T^k(i \circ \beta_k)$ (this is a finite sum). For $k \neq 0$, β_k is just the linking coefficient $L(\tilde{g}, T^k\tilde{g}) \in \pi_p(S^q)$.

From Remark (3) of §1, the symmetry of the linking coefficients $L(\tilde{g}, T^k\tilde{g})$ and $L(T^k\tilde{g}, \tilde{g}) = L(\tilde{g}, T^{-k}\tilde{g})$ can be written as

$$[T^k(i \circ \beta_k), i] + [i \circ \beta_{-k}, T^k i] = 0;$$

i.e.,

$$[T^{-k}(i \circ \beta_{-k}), i] = -T^{-k}[T^k(i \circ \beta_k), i].$$

Hence

$$(T-1)\alpha = [\beta, i] = [i \circ \beta_0, i] + \sum_{k=1}^{\infty} (1 - T^{-k})[T^k(i \circ \beta_k), i].$$

Put $T = 1$ in the above formula. We obtain $[i \circ \beta_0, i] = 0$. Hence, since $T - 1$ is injective, we get that

$$\alpha = \sum_{k=1}^{\infty} T^{-k}(1 + T + \cdots + T^{k-1})[T^k(i \circ \beta_k), i].$$

Putting $T = 1$ gives

$$L(f, g) = \sum_{k=1}^{\infty} k[\beta_k \circ i, i].$$

Since β_k is stable (cf. [W]), this is equal to $[i, i] \circ \Sigma_{k=1}^{\infty} k\tilde{\beta}_k$, where $\tilde{\beta}_k$ is (up to sign) the iterated suspension of β_k.

4. Proofs of Theorems 4 and 8.

Proof of Theorem 8. Let $X = S^{n+2} \setminus S^k \cup_\alpha e^n$. Then X is stably homotopy equivalent to $S^k \cup_\alpha e^n$ (cf. [C or Ha]). Now $H^1(X) = H_n(S^k \cup_\alpha e^n) = \mathbf{Z}$. Thus there is a map $X \to S^1$ inducing an isomorphism on H_1. Hence stably, X retracts onto the bottom cell, and hence α is stably trivial.

Proof of Theorem 4. Put $K = \overline{S^m \setminus N(S^{m-2})}$, $K_X = \overline{S^m \setminus N(X)}$, $X = S^p$, where $N(S^{m-2})$, $N(X)$ are regular neighborhoods. Consider the diagram

$$\begin{tikzcd} S^1 \times S^{m-2} \arrow[r, "1 \times L"] \arrow[d] \arrow[dr, "\rho"] & S^1 \times K_X \arrow[d] \\ S^1 \wedge S^{m-2} \arrow[r, "\Sigma L"] & S^1 \wedge K_X \end{tikzcd}.$$

The map ρ sends $S^1 \vee S^{m-2}$ to the basepoint. We must show that ρ is homotopic to a constant map rel $S^1 \vee S^{m-2}$.

In fact, it suffices to show that ρ is homotopic to a constant by a homotopy only rel S^1 and not rel $S^1 \vee S^{m-2}$. For if h_t is a homotopy (rel S^1) of ρ to a constant, then $g_t = h_{1-t}|_{S^{m-2}} \circ p_2$ (where p_2: $S^1 \times S^{m-2} \to S^{m-2}$ is the projection) is a self-homotopy (rel S^1) of the constant map $S^1 \times S^{m-2} \to S^1 \wedge K_X$, which on S^{m-2}

is inverse to h_t. The combined homotopy

$$H_t = \begin{cases} h_{2t}, & t \leqslant 1/2, \\ g_{2t-1}, & t \geqslant 1/2, \end{cases}$$

may therefore be homotoped to a homotopy (rel $S^1 \vee S^{m-2}$) of ρ to a constant.

Now $N(S^{m-2}) = D^2 \times S^{m-2}$ (we are assuming $S^{m-2} \to S^m$ is locally flat). Hence $\partial K = \partial N(S^{m-2}) = S^1 \times S^{m-2}$. Suppose we show that ρ can be extended to a map $K \to S^1 \wedge K_X$. Then since $S^1 \to K$ is a homology equivalence, and $S^1 \to S^1 \wedge K_X$ is the constant map, it follows that $K \to S^1 \wedge K_X$ (and hence $S^1 \times S^{m-2} \to S^1 \wedge K_X$) is homotopic (rel S^1) to a constant map.

Thus it suffices to extend ρ to a map $K \to S^1 \wedge K_X$. To do this, note that, since $H^1(K) = H^1(S^1 \times S^{m-2}) = H^1(S^1) = \mathbf{Z}$ and since S^1 is a $K(\mathbf{Z}, 1)$, the projection $S^1 \times S^{m-2} \to S^1$ may be extended to a map φ: $K \to S^1$. Moreover, since (by hypothesis) $H_1(X) \to H_1(K)$ is trivial, we may assume that φ sends $N(X)$ to the basepoint.

Now the map $S^1 \times S^{m-2} \to S^{m-2} \overset{L}{\subset} K_X$ extends to $K \cap K_X \to K_X$ (which, up to homotopy, is just the inclusion). Thus the map $S^1 \times S^{m-2} \xrightarrow{1 \times L} S^1 \times K_X$ extends to a map $K \cap K_X \to S^1 \times K_X$ such that the composite $K \cap K_X \to S^1 \wedge K_X$ sends $\partial N(X)$ to the basepoint. We may thus extend ρ to all of K by sending $N(X)$ to the basepoint.

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