PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 2, February 1986

METRIZABILITY OF GENERAL ANR

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Dedicated to Professor Keiô Nagami on the occasion of his 60th birthday

ABSTRACT. We show that every nonmetrizable ANR($\mathscr{P}$) contains a copy of a Tychonoff cube of uncountable weight. Hence, every finite dimensional ANR($\mathscr{P}$) is metrizable, and every ANR($\mathscr{P}$), each point of which is a G_δ-set, is metrizable, where $\mathscr{P}$ denotes the class of all paracompact p-spaces.

1. Introduction. In this paper we shall discuss ANR($\mathscr{F}$) spaces for some class $\mathscr{F}$ of topological spaces *beyond* that of metrizable spaces (that is, $\mathscr{F}$ contains nonmetrizable members).

One of the fundamental problems in the theory of ANR($\mathscr{F}$)'s is to find *useful criterions* to determine whether or not a given space is an ANR($\mathscr{F}$). In this aspect we believe that Ščepin's result **[7]** and its elementary but useful generalization **[8]** for the class of compact spaces are important. In the preceding note **[11]** we have extended his former result for the class of σ-locally compact, paracompact p-spaces.

In this note we shall extend these results for the class $\mathscr{P}$ of *paracompact p-spaces*. (They are precisely the perfect preimages of metric spaces, or equivalently, those spaces which are homeomorphic to a closed subspace of the product of a metric space and a compact space.) Then, the purpose of this paper is to show the following:

THEOREM. *Every nonmetrizable* ANR($\mathscr{P}$) *contains a copy of a Tychonoff cube of uncountable weight.*

COROLLARY 1. *Every finite dimensional* ANR($\mathscr{P}$) *is metrizable.*

The above corollary gives a positive solution to the problem of Telgársky (see **[11**, Problem 1]). We also prove

COROLLARY 2. *Every* ANR($\mathscr{P}$), *each point of which is a G_δ-set, is metrizable. In particular, every first-countable* ANR($\mathscr{P}$) *is metrizable.*

In this paper all maps are assumed to be continuous. For the undefined terminology refer to **[1, 3, 6]**.

Received by the editors October 16, 1984 and, in revised form, March 1, 1985. This material was presented at the 21th General Topology Symposium held at Tokyo Gakugei University, Tokyo, on June 5, 1985.

1980 *Mathematics Subject Classification*. Primary 54C55; Secondary 54B35, 54D18, 54F15, 54F45.

Key words and phrases. ANR, paracompact p-space, generalized Peano continuum.

The author expresses his thanks to Professor M. Tsuda, the referee, and the editor for their careful reading of an early version of this paper and for their valuable comments. He also would like to express deep appreciation to Professor Telgársky for raising the problem considered here and for his constant attention to this paper **[9, 10]**.

2. Proofs of our results. We start with the proof of the main Theorem. Let X be an ANR($\mathscr{P}$) which contains no copy of a Tychonoff cube of uncountable weight. Then, it suffices to show that X is metrizable. We can assume **[5]** that X is a closed subspace of a product space $I^\tau \times T$ of a Tychonoff cube I^τ ($\tau = w(X)$) and a metric space T. Let $r\colon U_0 \to X$ be a retraction for some *cozero-set* neighborhood U_0 of X in $I^\tau \times T$. Since T is metrizable, there exists a perfect map q from some strongly zero-dimensional metric space Y onto T. Take a cozero-set neighborhood U' of X in $I^\tau \times T$ such that $X \subset U' \subset \overline{U}' \subset U_0$, where $\overline{U}'$ denotes the closure of U'. Put

$$p = 1_{I^\tau} \times q, \qquad Z_0 = p^{-1}(U_0), \qquad Z = p^{-1}(U'),$$
$$f_0 = r \cdot p|Z_0\colon Z_0 \to X \quad \text{and} \quad f = f_0|Z.$$

Let $\{\mathscr{U}_i\}_{i=1}^\infty$ be a sequence of disjoint clopen covers of Y which satisfies the condition that $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$ and that the mesh of U_i is less than $1/i$ for every i. For each $\mathscr{U} \in \mathscr{U}_i$ take a point $s(U)$ of U. Then put

$$L(U) = Z_0 \cap (I^\tau \times \{s(U)\}).$$

Since Z_0 is a cozero-set, $L(U)$ is a countable union of Tychonoff cubes of weight τ. Therefore, its image $f_0(L(U))$ is a Lindelöf σ-space, since X cannot contain nonmetrizable generalized Peano continua **[8]**. Hence, by **[2**, Remark following the proof of Theorem 2] there exist a countable set $\tau(U) \subset \tau$ and a map $f_U\colon \pi_{\tau(U)}(Z_0) \to X$, where $\pi_{\tau(U)}\colon I^\tau \times Y \to I^{\tau(U)}$ is a natural projection, which satisfy

(1) $f_0(z) = f_U \circ \pi_{\tau(U)}(z)$ for every $z \in L(U)$;

(2) $\tau(U) \supset \tau(V)$, where V is the unique element of $\mathscr{U}_{i-1}$ which contains U.

Next, for each i we define a metric space M_i as

$$M_i = \oplus\{I^{\tau(U)}\colon U \in \mathscr{U}_i\}.$$

For each $z = (t, y) \in Z$, where $t \in I^\tau$, $y \in Y$, define

$$\varphi_i(z) = i_U \circ \pi_{\tau(U)}(t),$$

where $i_U\colon I^{\tau(U)} \to M_i$ is a natural inclusion map and U is a unique element of $\mathscr{U}_i$ containing y. Then, $\varphi_i\colon Z \to M_i$ is a continuous map. We shall now consider another topology in the set Z, coarser than the original one, which is defined by declaring

(3) a set $U \subset Z$ is open if for every $x \in U$ there exist a natural number i and an open set W of M_i such that $x \in \varphi_i^{-1}(W) \subset U$.

From (2) this topology is well defined. The set Z with this topology is denoted by Z'. Next, we show that

(4) the map $f'\colon Z' \to X$, defined by $f'(z) = f(z)$ for $z \in Z'$, is continuous.

Let W be any open set in X and take an arbitrary point $z = (t, y)$ in $f'^{-1}(W) = f^{-1}(W)$, where $t \in I^\tau$, $y \in Y$. Then, from (1) we can show without difficulty that

(5) $f_0(t, y) = f_0(t', y)$ if $\pi_\nu(t) = \pi_\nu(t')$, where $\nu = \bigcup_{i=1}^\infty \tau(U_i)$ and U_i is the unique element of $\mathscr{U}_i$ containing y.

Since f is continuous, there exist open sets $G \subset I^\tau$ and $U \subset Y$ with

(6) $z \in G \times U \subset \overline{G} \times U \subset f^{-1}(W)$.

Then, for some i there exist open sets $O' \subset I^{\tau(U_i)}$ and $H' \subset I^\mu$, where $\mu = \tau - \nu$, such that

$$U_i \subset U, \quad O = \pi_{\tau(U_i)}^{-1}(O'), \quad H = \pi_\mu^{-1}(H'), \qquad t \in O \cap H \subset G.$$

For each $j \geqslant i$ put

$$K_j = \zeta_i^{-1}(O') \cap \left(I^\tau \times U_j\right).$$

Note that $z \in K_j$ for each $j \geqslant i$. We show that there exists a $k \geqslant i$ such that

(7) $f(K_k) \subset W$.

On the contrary, assume that there exists a sequence $\{z_j = (t_j, y_j)\}_{j \geqslant i}$ of points in Z such that

(8) $z_j \in K_j$ and $f(z_j) \in X \setminus W$.

Then, taking a subsequence if necessary, we can assume that the sequence $\{\pi_\nu(t_j)\}$ converges to some point s_0 of the compact *metric* space $\overline{\pi_\nu(G)}$. Let s be an accumulation point of the sequence $\{t_j\}$ in I^τ. Note that $\pi_\nu(s) = s_0$. Put $z_\infty = (s, y)$. Then, $z_\infty \in \overline{Z} \subset Z_0$. Since f_0 is continuous and $f = f_0|Z$, we have from (8) that $f_0(z_\infty) \in X \setminus W$. On the other hand, put

$$w = (t', y), \quad \text{where } \pi_\nu(t') = s_0 \quad \text{and } \pi_\mu(t') = \pi_\mu(t).$$

Then, $w \in \overline{G} \times U$, and hence from (5) and (6) we have $f_0(z_\infty) = f_0(s, y) = f_0(t', y) = f(t', y) \in W$. This is a contradiction. Hence, (7) holds, and then, (4) follows from (3) and (7). Put

$$\varphi = \Delta\varphi_i \colon Z \to \prod_{i=1}^\infty M_i \quad \text{and} \quad Z^* = \varphi(Z).$$

Then, $\varphi' \colon Z' \to Z^*$, defined by $\varphi'(z) = \varphi(z)$ for $z \in Z'$, is continuous, and is a quotient map from (3). Note that from (4) $f'(z) = f'(z')$ if $\varphi'(z) = \varphi'(z')$. Hence, there exists a map $f^* \colon Z^* \to X$ satisfying $f' = f^* \circ \varphi'$. Note that $\varphi \colon Z \to Z^*$ is the composition of the identity map $i \colon Z \to Z'$ and the natural quotient map $\varphi' \colon Z' \to Z^*$. Hence, $f = f^* \circ \varphi$. Put

$$X^* = \varphi\left(p^{-1}(X)\right) \quad \text{and} \quad g = f^* |X^* \colon X^* \to X.$$

We shall show that g is a perfect map. For every $x \in X$ we have $g^{-1}(x) = \varphi(p^{-1}(x))$. Hence, $g^{-1}(x)$ is compact, since p is perfect. Next, we show that g is a closed map. Let F be any closed set of X^*. Put

$$H = \varphi^{-1}(F) \cap p^{-1}(X).$$

Then, $g(F) = f^* \circ \varphi(H) = f(H) = r \circ p(H) = p(H)$. Hence, $g(F)$ is closed in X, since p is perfect and H is closed in $p^{-1}(X)$. Therefore, X is metrizable, since g is perfect and X^* is metrizable. That completes the proof of the Theorem.

PROOF OF THE COROLLARIES. Let X be a nonmetrizable ANR($\mathscr{P}$). Then, by our main theorem, X contains a topological copy of a Tychonoff cube I^{ω_1}, where ω_1 is the first uncountable cardinal. Therefore, X is infinite dimensional, and each point in $I^{\omega_1} \subset X$ is not a G_δ-set. That completes the proof of the corollaries.

REMARK. One of the main points of our theorem is that retractions are not necessarily *closed* in our cases. It seems that the assumption of the existence of closed retraction is a quite restricted one. For example, there are no *closed* retractions from the Euclidean plane R^2 onto its x-axis $R \times \{0\}$. For the more special case when X is an AR with a closed retraction our results follow from [**4**, Theorem 1.1] and [**12**, Corollary 2], respectively.

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