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# COMPLETELY SEMISIMPLE SEMIGROUPS AND EPIMORPHISMS

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ABSTRACT. It is proved that a completely semisimple semigroup $U$ cannot be properly epimorphically embedded in another semigroup if $U$ has no infinite chain of $\mathcal{J}$-classes.

Let $U, S$ be semigroups with $U \subseteq S$. Following Howie and Isbell [**8**] we say $U$ *dominates* an element $d \in S$ if for every semigroup $T$ and all homomorphisms $f: S \to T$ and $g: S \to T$, $uf = ug$ for all $u \in U$ implies $df = dg$. The set of all elements of $S$ dominated by $U$ is called the *dominion* of $U$ in $S$ and is denoted by $\operatorname{Dom}(U, S)$. It is easily verified that $\operatorname{Dom}(U, S)$ is a subsemigroup of $S$ containing $U$. We say $U$ is *closed* in $S$ if $\operatorname{Dom}(U, S) = U$ and $U$ is *absolutely closed* if $U$ is closed in every containing semigroup $S$. A semigroup $U$ is *saturated* if $\operatorname{Dom}(U, S) \neq S$ for every properly containing semigroup $S$.

Let $f: S \to T$ be a morphism of semigroups. Then $f$ is an *epimorphism* (epi for short) if for every pair of morphisms $g: T \to V$ and $h: T \to V$, $fg = fh$ implies $g = h$. It is easy to check that a morphism $f: S \to T$ is epi if and only if the inclusion $i: f(S) \to T$ is epi, and if $U \subseteq S$ then $\operatorname{Dom}(U, S) = S$ if and only if $i: U \to S$ is epi, whereupon we say that $U$ is *epimorphically embedded* in $S$.

Most results concerning semigroup dominions and epimorphisms are based on the following result.

RESULT 1 (ISBELL'S ZIGZAG THEOREM [**7**, THEOREM 2.13]). *Let $U$ be a subsemigroup of a semigroup $S$ and let $d \in S$. Then $d \in \operatorname{Dom}(U, S)$ if and only if $d \in U$ or there is a series of factorizations of $d$ as follows:*

$$d = u_0 y_1 = x_1 u_1 y_1 = x_1 u_2 y_2 = x_2 u_3 y_2 = \cdots = x_m u_{2m-1} y_m = x_m u_{2m},$$

*where $m \geq 1$, $u_i \in U$, $x_i, y_i \in S$ with $u_0 = x_1 u_1$, $u_{2i-1} y_i = u_{2i} y_{i+1}$, $x_i u_{2i} = x_{i+1} u_{2i+1}$ $(1 \leq i \leq m-1)$, $u_{2m-1} y_m = u_{2m}$.*

Such a series of factorizations is called a *zigzag* in $S$ over $U$ with *value $d$*, *length $m$* and *spine* $u_0, u_1, \ldots, u_{2m}$.

Semigroup dominions can also be expressed in terms of special semigroup amalgams [**7**, Theorem 2.3]. In particular $U$ is absolutely closed if and only if every special amalgam with core $U$ is strongly embeddable in a semigroup.

Most notable results in this area have been to the effect that certain classes consist entirely of absolutely closed or of saturated semigroups.

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For example, it was shown by Howie and Isbell [**8**] that right simple semigroups, finite monogenic semigroups and inverse semigroups are absolutely closed. Scheiblich and Moore [**10**] showed that the total transformation semigroup, $\mathcal{T}_x$, is absolutely closed; a result also proved in Shoji [**11**] and by Hall [**1**] whose proof also works for the semigroup of partial transformations on a set.

Howie and Isbell [**8**] proved that any commutative semigroup that satisfies $M_{\mathcal{J}}$, the minimum condition on $J$ classes, is saturated and the same is true of any finite permutative semigroup (a semigroup is permutative if it satisfies a nontrivial permutation identity) [**6**]. Another result due to the author [**3**] is that generalized inverse semigroups are saturated (regular semigroups whose idempotents form a normal subband). Finally, a strong result due to Hall and Jones [**2**] is that every completely semisimple semigroup with a finite number of $\mathcal{J}$-classes is saturated; in particular this includes all completely [0-] simple and finite regular semigroups. We prove a stronger result below.

On the other hand the $2 \times 2$ rectangular band is not absolutely closed [**8**] and the 3-element null semigroup can have an infinite dominion [**1**]. Furthermore there are semigroups of each of the following types which are not saturated; commutative cancellative semigroups (the injection of the natural numbers into the integers under addition provides an example), subsemigroups of finite inverse semigroups [**9**], commutative, periodic semigroups [**4**], and bands [**5**]. Indeed recently Trotter [**12**] has constructed a band with a properly epimorphically embedded subband.

The following theorem was announced at the Marquette Semigroup Conference in 1984. The result will appear in the Proceedings of that conference.

THEOREM 2. *A completely semisimple semigroup $U$ is saturated if it has no infinite chain of $J$-classes.*

Before we give the proof we make a couple of technical observations concerning zigzags. Let $Z$ be a zigzag as described in Result 1. If the length of $Z$ is minimum amongst all those zigzags in $S$ over $U$ with value $d$, then $x_i, y_i \in S\backslash U$ for all $i = 1, 2, \ldots, m$. Next suppose that $U$ is properly epimorphically embedded in $S$, and that two successive lines of a zigzag $Z$ of minimum length $m$, with value $d \in S\backslash U$, are $x_i u_{2i} y_{i+1} = x_{i+1} u_{2i+1} y_{i+1}$ $(1 \leq i \leq m-1)$. Then, since $y_{i+1} \in S\backslash U$ we have, by the Zigzag Theorem, $y_{i+1} = a y'_{i+1}$ for some $a \in U$, $y'_{i+1} \in S\backslash U$. We may then construct a modified zigzag $Z'$, with value $d$, where the two given lines are replaced by $x_i u_{2i} a y'_{i+1} = x_{i+1} u_{2i+1} a y'_{i+1}$ because the necessary equalities are provided by $u_{2i-1} y_i = u_{2i} a y'_{i+1}$, $x_i u_{2i} a = x_{i+1} u_{2i+1} a$ and $u_{2i+1} a y'_{i+1} = u_{2i+2} a y'_{i+1}$. A similar remark also applies to the two initial lines of the zigzag: $u_0 y_1 = x_1 u_1 y_1$. We call the process of passing from $Z$ to $Z'$ "expansion of $Z$ at $y_{i+1}$ via the factorization $y_{i+1} = a y'_{i+1}$". Of course we can perform this modification for every $y_i$ if we choose. There is a dual comment which applies to the $x_i$ which is also true.

It will be understood throughout that Green's relations are relations in $U$.

PROOF OF THEOREM 2. Suppose that $U$ is a completely semisimple semigroup with no infinite chain of $\mathcal{J}$-classes properly epimorphically embedded in $S$.

Take $d \in S\backslash U$. Let $\mathcal{C}_d$ be the collection of all $\mathcal{J}$-classes $J$, such that some $u \in J$ is the first spine member of some zigzag of minimum length in $S$ over $U$ with value

$d$. Denote the collection of all minimal members of $\mathcal{C}_d$ by $\overline{\mathcal{C}}_d$ and put

$$J_R = \bigcup_{d \in S\setminus U} \overline{\mathcal{C}}_d.$$

The dual collection of $\mathcal{J}$-classes will be denoted by $J_L$.

Take $J$ to be a maximal member of $J_R \cup J_L$. Without loss we assume that $J \in J_R$, so that there exists $d \in S\setminus U$ such that $d = u_0 y_1$ say, is the first line of the zigzag $Z$ of minimum length $m$ in $S$ over $U$ with value $d$, $u_0 \in J$, and if $d = uy$ is the first line of another such zigzag, then $J_u \not< J$. The maximality condition on $J$ guarantees that the $\mathcal{J}$-class of the first spine member of $Z$ is invariant under any expansion. We may also assume that the $\mathcal{J}$-class corresponding to each particular spine member is invariant under expansion at $x_i$ or $y_i$ $(1 \leq i \leq m)$, because we may expand $Z$ at each $x_i$ or $y_i$ until the $\mathcal{J}$-class of each spine member is fixed under any further expansions, which must occur as $U$ satisfies the descending chain condition on $\mathcal{J}$-classes. We shall assume that this process has been carried out.

We show that $u_0, u_1, \ldots, u_{2m-1} \in J$. We have $u_0 \in J$, so assume inductively that $u_0, u_1, \ldots, u_{2i} \in J$ $(0 \leq i \leq m-1)$. We first prove that $J \leq J_{u_{2i+1}}$. First if $i = 0$ the equality $u_0 = x_1 u_1$ implies that $u_0 = u_0 u_1' u_1$ $(u_1' \in V(u_1))$, whence $J \leq J_{u_1}$. If $i > 0$ we factorize $x_i$ as $x_i' a_i$, where $a_i \in J_L$. We obtain

$$J = J_{u_{2i}} = J_{a_i u_{2i}} \leq J_{a_i}, \tag{1}$$

where the second equality is justified by the invariance of the $\mathcal{J}$-class under expansion. However, $J < J_{a_i}$ is impossible because of the maximality condition on $J$, and thus we have equality throughout in (1) and $a_i$, $u_{2i}$, $a_i u_{2i} \in J$.

Since $R_{a_i u_{2i}} \leq R_{a_i}$, $a_i u_{2i} \mathcal{J} a_i$ and $U$ is completely semisimple, it follows that $R_{a_i u_{2i}} = R_{a_i}$. Hence $a_i = a_i u_{2i} t_i$ for some $t_i \in U^1$, whence we have

$$x_i = x_i' a_i = x_i' a_i u_{2i} t_i = x_i u_{2i} t_i = x_{i+1} u_{2i+1} t_i.$$

Consider the zigzag $Z'$ which results from expansion of $Z$ at $x_i$ via the factorization $x_i = x_{i+1} u_{2i+1} t_i$. Since the $\mathcal{J}$-class of each spine member of $Z$ is invariant under any expansion we obtain

$$J = J_{u_{2i}} = J_{u_{2i+1} t_i u_{2i}} \leq J_{u_{2i+1}},$$

as asserted.

Next factorize $y_{i+1}$ as $b_{i+1} y_{i+1}'$ with $b_{i+1} \in J_R$. This gives

$$J \leq J_{u_{2i+1}} = J_{u_{2i+1} b_{i+1}} \leq J_{b_{i+1}},$$

and once again strict inequality is impossible because of the maximality condition on $J$. Therefore $J = J_{u_{2i+1}}$.

The dual argument now establishes that given $u_0, u_1, \ldots, u_{2i+1} \in J$ $(0 \leq i \leq m-2)$, then $u_{2i+2} \in J$. Thus we have proved that $u_0, u_1, \ldots, u_{2m-1} \in J$. Arguing as before we obtain $y_m = b_m y_m'$ with $b_m \in J$ and $b_m = s_m u_{2m-1} b_m$ for some $s_m \in U^1$. But then

$$y_m = b_m y_m' = s_m u_{2m-1} b_m y_m' = s_m u_{2m-1} y_m = s_m u_{2m} \in U^1$$

which is a contradiction as $y_m \in S\setminus U$. This completes the proof.

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