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A NOTE ON MINIMAL MODULAR SYMBOLS

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ABSTRACT. For any arithmetic group, a set of geometrically-defined cohomology classes is constructed which spans the cohomology of the group with rational coefficients in the highest nonvanishing dimension thereof.

This note gives a geometrical spanning set S for the cohomology of an arithmetic group in the highest nonvanishing dimension. To do this, I generalize the first three sections of [**1**]. Unfortunately, I know of no way to generalize the algorithm in §4 of [**1**]. This means that while the S I construct is infinite, and although a priori the cohomology is finite dimensional, I cannot identify a finite generating set inside S, except for the cases covered in [**1**].[1]

Let G be a semisimple algebraic group defined over $\mathbf{Q}$, K a maximal compact subgroup of $G(\mathbf{R})$, $X = G(\mathbf{R})/K$, and Γ an arithmetic subgroup of $G(\mathbf{Q})$. The action of $G(\mathbf{R})$, and hence of Γ, on X is on the left. It is well known that $H^*(\Gamma, \mathbf{Q}) \simeq H^*(X/\Gamma, \mathbf{Q})$, and by [**2**] that $H^*(\Gamma, \mathbf{Q}) = 0$ for $* > N$, where

$$N = \dim X - \operatorname{rank}_{\mathbf{Q}}(G).$$

To construct $S \subset H^N(X/\Gamma, \mathbf{Q})$ we proceed as follows: Let T be a maximal $\mathbf{Q}$-split torus of G. Without loss of generality, we may assume that T is stable under the Cartan involution of G corresponding to K. Set $A = T(\mathbf{R})^0$. Thus $A \simeq (\mathbf{R}_+^\times)^l$, where $l = \operatorname{rank}_{\mathbf{Q}}(G)$.

Now let $e \in X$ be the base-point corresponding to K. Let $\overline{X}$ be the Borel-Serre bordification of X [**2**]. From the construction of $\overline{X}$ given in [**2**], it is easy to verify the following:

LEMMA 1. *The closure Z of Ae in $\overline{X}$ is homeomorphic to a ball of dimension l. The boundary of Z lies in $\partial\overline{X}$.*

Now fix an orientation on Z.

LEMMA 2. *The fundamental class of ∂Z freely generates $H_{l-1}(\partial\overline{X}, \mathbf{Z})$ as a $\mathbf{Z}[U(\mathbf{Q})]$-module, where U is the unipotent radical of a minimal $\mathbf{Q}$-parabolic subgroup P of G containing T.*

The proof of this lemma is essentially contained in §8 of [**2**]. The central object is the Tits building B of the $\mathbf{Q}$-group G. The homotopy equivalence $g_x: B \to \partial\overline{X}$ constructed in [**2**, 8.4.3] maps one of the apartments of B homeomorphically onto ∂Z, if we choose x to be the identity coset e in X. Call this distinguished apartment A_0. Then choosing an $(l-1)$-dimensional simplex s of A_0 corresponds to the choice of P in the lemma. Each apartment is homeomorphic to an $(l-1)$-sphere and B

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is homotopy equivalent [**2**, 8.5.2] to the wedge of all the apartments of B that contain s. Hence $\partial\overline{X}$ is homotopy-equivalent to a wedge of spheres, and [**2**, 8.6.8] implies that $H_{l-1}(\partial\overline{X}, \mathbf{Z})$ is a free $\mathbf{Z}[U(\mathbf{Q})]$-module of rank 1, generated by the fundamental class of $g_e(A_0) = \partial Z$.

We shall use the notation $[I]$ for the fundamental class of ∂Z in $H_{l-1}(\partial\overline{X}, \mathbf{Z})$, where I stands for the identity in $G(\mathbf{Q})$. More generally, if M is any element of $G(\mathbf{Q})$, set $[M]$ to be $M \cdot [I]$, i.e. the fundamental class of $\partial(MZ)$. We call $[M]$ a universal minimal modular symbol. ("Minimal" refers to the fact that it is associated with a minimal $\mathbf{Q}$-parabolic subgroup of G.)

Now let π denote the canonical projection of $\overline{X}$ onto $\Gamma\backslash\overline{X}$. From now on, assume $l \geq 1$. Since $\overline{X}$ is contractible, $H_{l-1}(\partial\overline{X}, \mathbf{Z})$ is isomorphic to $H_l(X, \partial\overline{X}, \mathbf{Z})$. Now define $[M]_\Gamma$, for any M in $G(\mathbf{Q})$, to be the image of $[M]$ in $H_l(\Gamma\backslash\overline{X}, \partial(\Gamma\backslash\overline{X}), \mathbf{Z})$ under the composition

$$H_{l-1}(\partial\overline{X}, \mathbf{Z}) \xrightarrow{\sim} H_l(\overline{X}, \partial\overline{X}, \mathbf{Z}) \xrightarrow{\pi_*} H_l(\Gamma\backslash\overline{X}, \partial, \mathbf{Z}).$$

It follows from Lemma 2.7 of [**4**] that $\pi(M(Ae))$ is a submanifold (with boundary) of $\Gamma\backslash\overline{X}$. Then $[M]_\Gamma$ is its fundamental class.

THEOREM. *If Γ is torsion-free, the modular symbols $[M]_\Gamma$ as M runs through $U(\mathbf{Q})$, or a fortiori through $G(\mathbf{Q})$, generate $H_l(\Gamma\backslash\overline{X}, \partial, \mathbf{Z})$.*

PROOF. This follows from the fact that π_* is surjective. Although the proof of this fact is done in a special case in [**1**], the argument carries over verbatim.

REMARK. By Poincaré duality, $H_l(\Gamma\backslash\overline{X}, \partial, \mathbf{Z})$ is naturally isomorphic to $H^N(\Gamma\backslash\overline{X}, \mathbf{Z})$ when Γ is torson-free, so we may take the promised set S to be the set of currents corresponding to the modular symbols $[M]_\Gamma$, M in $U(\mathbf{Q})$.

If Γ is not torsion-free, the theorem and remark remain true if $\mathbf{Z}$-coefficients are replaced by $\mathbf{Q}$-coefficients (cf. remark, p. 246 of [**1**]).

EXAMPLE. If F is any finite extension of $\mathbf{Q}$, let G be the $\mathbf{Q}$-group such that $G(\mathbf{Q}) = \mathrm{SL}(n, F)$, $n \geq 2$. Then

$$G(\mathbf{R}) = \mathrm{SL}(N, F \otimes_{\mathbf{Q}} \mathbf{R}).$$

View $\mathbf{R}$ as embedded in the natural way in $F \otimes \mathbf{R}$. Then A may be taken to be the diagonal matrices in $G(\mathbf{R})$ with entries in $\mathbf{R}_+^\times$, and $l = n - 1$. In case the ring of integers in F is Euclidean, the algorithm in §4 of [**1**] shows how to reduce S to a finite spanning set. But otherwise, a general algorithm for this problem is unknown and would be of great interest.

REMARKS. (1) Nonminimal modular symbols may be defined in other dimensions, as in [**4**]. For these symbols, no such assertion that they generate the cohomology in their dimension is known to be true, nor is any counterexample known to me.

(2) The set $S = \{[M]_\Gamma : M \in U(\mathbf{Q})\}$ depends upon the choices of T, K and U. We may enlarge S to obtain a set which does not depend on these choices, namely $\{[M]_\Gamma : M \in G(\mathbf{Q})\}$.

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