PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 3, March 1986

ANALYTIC FUNCTIONALS AND THE BERGMAN PROJECTION ON CIRCULAR DOMAINS

PAUL ZORN

ABSTRACT. A property of the Bergman projection associated to a bounded circular domain containing the origin in $\mathbf{C}^N$ is proved: Functions which extend to be holomorphic in large neighborhoods of the origin are characterized as Bergman projections of smooth functions with small support near the origin. For certain circular domains D, it is also shown that functions which extend holomorphically to a neighborhood of $\overline{D}$ are precisely the Bergman projections of smooth functions whose supports are compact subsets of D. Two applications to proper holomorphic mappings are given.

This paper treats properties of the Bergman projection on certain domains in $\mathbf{C}^N$. We denote by $L^2(D)$ the space of functions which are square integrable with respect to Lebesgue measure on a domain D, and by $H_2(D)$ the (Bergman) space of holomorphic functions in $L^2(D)$. The orthogonal projection $P: L^2(D) \to H_2(D)$ is called the *Bergman projection*. The *Bergman kernel* is the integral kernel $k(z,w)$ such that, for all f in $L^2(D)$,

$$Pf(z) = \int_D f(w)k(z,w)\,dw.$$

An *analytic functional* on a domain D in $\mathbf{C}^N$ is a continuous linear functional on $\mathbf{O}(D)$, the space of holomorphic functions on D, with the topology of uniform convergence on compact subsets of D. The connection with the Bergman projection is

LEMMA 1. *Let D be a domain in $\mathbf{C}^N$, f in $H_2(D)$, and U an open, relatively compact subset of D. There exists u in $\mathbf{C}_0^\infty(U)$ with $Pu = f$ if and only if, for some compact $K \subset U$ and some constant $C > 0$,*

$$\left|\int_D \overline{f}g\right| \leq C\|g\|_K \tag{1}$$

holds for all g in $H_2(D)$, where $\|g\|_K = \sup\{|g(z)|: z \text{ in } K\}$, and the integral is with respect to Lebesgue measure on $\mathbf{C}^N$.

PROOF. Consider the linear functional $T_f: H_2(D) \to \mathbf{C}$ given by

$$T_f(g) = \langle g, f\rangle = \int_D \overline{f}g.$$

If (1) holds, then T_f extends to a continuous linear functional on $\mathbf{C}(K)$, the space of continuous functions on K. Hence T_f is represented by a measure μ supported

Received by the editors April 26, 1984 and, in revised form, February 25, 1985.
1980 *Mathematics Subject Classification*. Primary 32A07, 32H10; Secondary 46E20.
Key words and phrases. Bergman projection, circular domain, analytic functional.

on K:

$$T_f(g) = \int_K g\, d\mu.$$

The measure μ is smoothed by convolution with a radially symmetric function of small support. The converse assertion follows easily from the orthogonality property of the Bergman projection. For details, see [**6**].

A *circular domain* in $\mathbf{C}^N$ is one invariant under the one-parameter family $z \to e^{it}z$ of biholomorphisms of $\mathbf{C}^N$. Let us recall several properties of such domains. First, if D is bounded and contains the origin, then there exists an orthonormal basis $\{p_1, p_2, \ldots\}$ of $H_2(D)$ consisting of homogeneous polynomials.[1] From the formula

$$k(z,w) = \sum_{j=1}^{\infty} p_j(z)\overline{p_j(w)},$$

it follows that for z and w in D the identity $k(tz, w) = k(z, \bar{t}w)$ holds for all complex t for which it is meaningful. If $r < 1$ is such that $rD \subseteq D$, and w in rD is fixed, then $k(z,w) = k(z/r, rw)$ defines a holomorphic extension of $k(\cdot, w)$ to $(1/r)\overline{D}$.

A related property of the Bergman kernel on a bounded circular domain, not necessarily containing the origin, is that *for w in a fixed compact subset of D, $k(\cdot, w)$ extends holomorphically to a domain containing $\overline{D}$.* (See, e.g., [**1**].)

LEMMA 2. *Let D be a bounded circular domain. There is a larger domain D_1, containing D compactly, such that $\mathbf{O}(D_1)$ is dense in $H_2(D)$.*

REMARK. When D contains 0, the polynomials are dense in $H_2(D)$.

PROOF. Let U be any open set compactly contained in D. Let $S = \{Pu : u \in \mathbf{C}_0^\infty(U)\}$. Any g in S can be written $g(z) = \int_U u(w)k(z,w)\, dw$. By the extendibility property of $k(z,w)$ mentioned above, g extends holomorphically to a larger domain D_1 which depends only on U. Thus it suffices to show that S is dense in $H_2(D)$. Suppose f in $H_2(D)$ is orthogonal to S. Then

$$0 = \int_D \bar{f} Pu = \int_D \bar{f} u$$

holds for all u in $\mathbf{C}_0^\infty(U)$. Hence $f \equiv 0$ in U. This completes the proof.

The object of this paper is to prove the next two theorems:

THEOREM I. *Let $D \subseteq B$ $(= \{|z| < 1\})$ be a circular domain such that, for some $a > 0$, $aB = \{|z| < a\} \subseteq D$. Let r be real, $0 < r < a$. The following are equivalent:*

(1) *$f_{|D} = Pu$ for some u in $\mathbf{C}_0^\infty(rB)$,*

(2) *f in $\mathbf{O}(r^{-1}\overline{B})$.*

THEOREM II. *Let $D \subseteq B$ $(= \{|z| < 1\})$ be a circular domain with boundary bD of class $\mathbf{C}^1$ such that, for every p in bD, the complex line $\{tp : t \text{ in } \mathbf{C}\}$ intersects bD transversally. The following are equivalent:*

(1) *f is in $\mathbf{O}(\overline{D})$,*

(2) *$f_{|D} = Pu$ for some u in $\mathbf{C}_0^\infty(D)$.*

REMARK. Theorem I generalizes a result of Bell [**1**] for such domains: The restriction of a *polynomial* f to D can be written $f_{|D} = Pu$, where u is a smooth

[1] I am indebted to D. Barrett for pointing out this fact to me.

function which can be chosen to have support contained in any neighborhood of the origin. That (2) implies (1) in Theorem II means that D satisfies condition Q. (Smooth bounded strictly pseudoconvex domains with real-analytic boundary also satisfy condition Q. See [**2**].)

PROOF OF THEOREM I. (1) $\Rightarrow$ (2). If u is in $\mathbf{C}_0^\infty(rB)$, then

$$f(z) = Pu(z) = \int_D k(z,w)u(w)\,dw.$$

Since $k(\cdot,w)$ extends holomorphically to $r^{-1}\overline{B}$, so does f.

(2) $\Rightarrow$ (1). Choose $R < r$ so that f is in $\mathbf{O}(R^{-1}\overline{B})$, and define $\psi(z) = \chi_{RD}R^{-2N}f(R^{-2}z)$. Then

$$\begin{aligned} P\psi(z) &= R^{-2N}\int_{RD} k(z,w)f(R^{-2}w)\,dw = \int_D k(z,Rt)f(R^{-1}t)\,dt \\ &= \int k(Rz,t)f(R^{-1}t)\,dt = f(z), \end{aligned}$$

where the change of variable $w = Rt$ was made. If g is in $H_2(D)$, then by the orthogonality of the Bergman projection,

$$\left|\int_D \overline{f}g\right| = \left|\int_D \overline{P\psi}g\right| = \left|\int_{RD} \overline{\psi}g\right| \leq \text{volume}(RD)||\psi||_{RD}||g||_{RD} = C||g||_{RD}.$$

This, together with Lemma 1, completes the proof of Theorem I.

REMARK. The proof of Theorem II is similar to that of Theorem I except that more effort is required to verify the hypothesis of Lemma 1.

PROOF OF THEOREM II. That (2) $\Rightarrow$ (1) follows immediately from the fact that $k(z,w)$ extends (in z) across the boundary for w in a fixed compact subset of D. The remainder of this paper is a proof that (1) $\Rightarrow$ (2).

LEMMA 3. *Let $D \Subset \mathbf{C}^1$ be a domain with real-analytic boundary. If f is in $\mathbf{O}(\overline{D})$, then there exist $C > 0$ and a compactum $K \subset D$ so that, for all g in $H_2(D)$,*

$$\left|\int_D \overline{f}g\right| \leq C||g||_K.$$

PROOF. See [**6**, Theorem III.8]. See also [**2**, Lemma 1], where a simple proof is given for domains in $\mathbf{C}^N$.

REMARK. Lemma 3 is false for every $\mathcal{C}^2$ but not real-analytic bounded domain D in $\mathbf{C}^1$. (See [**6**, Theorem IV.3].) The situation in $\mathbf{C}^N$ is quite different: The domains in Theorem II can be $\mathbf{C}^2$ yet far from smooth.

To continue the proof of the theorem, we wish to find $C > 0$ and $K \subseteq D$ so that inequality (1) of Lemma 1 holds for all g in $H_2(D)$. There exists a domain $D_1 \supset \overline{D}$ so that $\mathbf{O}(D_1)$ is dense in $H_2(D)$, so it suffices to show that (1) holds for all g in $\mathbf{O}(D_1)$. Since both sides of (1) are positively homogeneous in g, we may assume further that $||g||_{\tilde{D}} = 1$ for some domain $\tilde{D}$ with $D \Subset \tilde{D} \Subset D_1$. Let such g be called *admissible*; the set of admissible g is a normal family on D.

We shall use Fubini's theorem for differential forms [**5**, p. 210] to estimate the integral in (1). Let $\Pi\colon \mathbf{C}^N\backslash\{0\} \to \mathbf{C}P^{N-1}$ be the projection given in homogeneous coordinates on $\mathbf{C}P^{N-1}$ by $\Pi(z_1,\ldots,z_N) = [z_1\colon\cdots\colon z_N]$. Let ω be the fundamental

(1, 1)-form of the Fubini-Study metric on $\mathbf{C}P^{N-1}$. A computation (see [**6**, p. 382] for details) yields

$$(\Pi^*\omega)^{N-1} \wedge \partial\overline{\partial}(|z|^2) = (i/2\pi)^{N-1}(N-1)!|z|^{2-2N}\, dz_1 \wedge \overline{dz_1} \cdots dz_N \wedge \overline{dz_N}.$$

It follows that if H is any measurable function defined on D,

$$\begin{aligned}
(N-1)!\left(\frac{2}{i\pi^{N-1}}\right)&\int_D H\, dm_{2N} \\
&= (N-1)!\left(\frac{i}{2\pi}\right)^{N-1}\int_{D\setminus\{0\}} H\, dz_1 \wedge \overline{dz_1} \cdots dz_N \wedge \overline{dz_N} \\
&= \int_{\alpha\in \mathbf{C}P^{N-1}}\left\{\int_{\Pi^{-1}(\alpha)\cap D} H|z|^{2N-2}\partial\overline{\partial}(|z|^2)\right\}\omega^{N-1},
\end{aligned}$$

where dm_{2N} is Lebesgue measure on $\mathbf{C}^N$. For fixed α in $\mathbf{C}P^{N-1}$, fix u in $\Pi^{-1}(\alpha)$ with $|u| = 1$. Then $\Pi^{-1}(\alpha) = \{\tau u \colon \tau \text{ in } \mathbf{C}\setminus\{0\}\}$. Taking τ to be the coordinate on $\Pi^{-1}(\alpha)$ gives

$$\int_{\Pi^{-1}(\alpha)\cap D} H|z|^{2N-2}\partial\overline{\partial}(|z|^2) = \frac{2}{i}\int_{\alpha\cap D} H(\tau)|\tau|^{2N-2}\, dm_2(\tau),$$

where $dm_2(\tau)$ is Lebesgue measure on the complex line α.

We apply this formula in the case $H = \overline{f}g$, where f and g are as above. Let α_0 in $\mathbf{C}P^{N-1}$ be fixed. Since D is circular, $D_{\alpha_0} \equiv D \cap \Pi^{-1}(\alpha_0)$ is also circular—in particular, D_{α_0} has real-analytic boundary. Since the restrictions of f and g to D_{α_0} are holomorphic, we may conclude from Lemma 3 that

$$\begin{aligned}
\left|\int_{D_{\alpha_0}} \overline{f}g|\tau|^{2N-2}\, dm_2(\tau)\right| &= \left|\int_{D_{\alpha_0}} \overline{f\tau^{N-1}}g\tau^{N-1}\, dm_2(\tau)\right| \\
&\le C_{\alpha_0}\|g\tau^{N-1}\|_{D(\alpha_0,\varepsilon_{\alpha_0})} < N_{\alpha_0}\|g\|_{D(\alpha_0,1/N_{\alpha_0})}
\end{aligned}$$

holds for all admissible g and some integer N_{α_0}, where $D(\alpha_0,\varepsilon) = \{z \text{ in } D_{\alpha_0}\colon \operatorname{dist}(z, bD_{\alpha_0}) \ge \varepsilon\}$. The second inequality holds because $\overline{D}$ is compact.

The inequality

$$\left|\int_{D_\alpha} \overline{f}gr^{2N-2}\, dm_2\right| < N_{\alpha_0}\|g\|_{D(\alpha,1/N_{\alpha_0})} \tag{2}$$

is valid for $\alpha = \alpha_0$ and arbitrary admissible g. For a fixed g, both sides of (2) are continuous in α, so there is a neighborhood U_{α_0} of α_0 in $\mathbf{C}P^{N-1}$ such that (2) holds for all α in U_{α_0}.

Since the admissible g are uniformly equicontinuous on $\overline{D}$, the neighborhoods U_{α_0} can be chosen so that (2) holds for all admissible g and all α in U_{α_0}. Since $\mathbf{C}P^{N-1}$ is compact, finitely many such neighborhoods $U_{\alpha_1}, \ldots, U_{\alpha_k}$ form an open cover. Taking $M = \max\{N_{\alpha_1}, \ldots, N_{\alpha_k}\}$, we have

$$\left|\int_{D_\alpha} \overline{f}gr^{2N-2}\, dm_2\right| \le M\|g\|_{D(\alpha,1/M)} \tag{3}$$

for all α and all admissible g.

Since all the D_α meet bD transversally, $K = \bigcup_{\alpha \in \mathbf{C}P^{N-1}} D(\alpha, 1/M)$ is relatively compact in D. Using (3),

$$\begin{aligned}\left|\int_D \overline{f} g\right| &= \frac{\pi^{N-1}}{(N-1)!}\left|\int_{\mathbf{C}P^{N-1}} \left\{\int_{D_\alpha} \overline{f} g r^{2N-2}\, dm_2\right\} \omega^{N-1}\right| \\ &\leq \frac{\pi^{N-1}}{(N-1)!}\left|\int_{\mathbf{C}P^{N-1}} M\|g\|_K\, \omega^{N-1}\right| \\ &\leq \frac{\pi^{N-1}}{(N-1)!} M\|g\|_K \left|\int_{\mathbf{C}P^{N-1}} \omega^{N-1}\right|.\end{aligned}$$

By the Wirtinger theorem, $(1/(N-1)!)\int_{\mathbf{C}P^{N-1}} \omega^{N-1}$ is the (finite) volume of $\mathbf{C}P^{N-1}$. This completes the proof of Theorem II.

We conclude with two applications to proper holomorphic mappings.

COROLLARY 1. *Suppose that D_1 is a smooth bounded domain satisfying condition* Q, *and that D_2 is as in Theorem* II. *If f is a proper holomorphic mapping of D_1 onto D_2, then f extends to be holomorphic in a neighborhood of $\overline{D}_1$.*

PROOF. The proof is identical to that of Theorem 1 in [**2**], except that our Theorem II replaces Bell's Lemma 1.

It is interesting to compare Corollary 1 to Theorem 2 in [**3**]: A proper holomorphic mapping of one complete Reinhardt domain onto another extends to be holomorphic in a neighborhood of the first.

COROLLARY 2. *Let D_1 be a bounded strictly pseudoconvex domain with real-analytic boundary, and D_2 a bounded smooth simply connected circular domain which satisfies the transversality condition of Theorem* II. *If f is a proper holomorphic mapping of D_1 onto D_2, then f extends to a biholomorphism between larger domains containing $\overline{D}_1$ and $\overline{D}_2$.*

REMARK. Corollary 2 generalizes a result of Bell [**4**]: Let D_1 be as in Corollary 2, D_2 a smooth bounded pseudoconvex complete Reinhardt domain. If f is a proper holomorphic mapping of D_1 onto D_2, then f extends to be a *biholomorphism* of larger domains.

PROOF. The proof is the same as in [**4**], except that Fact 2 follows from Theorem II.

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DEPARTMENT OF MATHEMATICS, ST. OLAF COLLEGE, NORTHFIELD, MINNESOTA 55057