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REMARKS ON PETTIS INTEGRABILITY

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ABSTRACT. Characterizations of Pettis integrability, including the Geitz-Talagrand core theorem, are derived in an easy way.

The purpose of this note is to point out how a folklore result (Proposition 1) can be made the basis for relatively easy proofs of some recent results about Pettis integrability. Our notation follows Dunford and Schwartz [**1**].

Let $(\Omega, \Sigma, \lambda)$ be a complete probability space, and let X be a Banach space with continuous dual X^*. A function $f: \Omega \to X$ is *Dunford integrable* provided the composition $T(x^*) = x^*f$ is in $L^1(\lambda)$ for every x^* in X^*. In this case, it follows (from the closed graph theorem) that $T: X^* \to L^1(\lambda)$ is a bounded linear operator. Hence, for every g in $L^\infty(\lambda)$, the map φ_g, defined by

$$\varphi_g(x^*) = \int gT(x^*)\,d\lambda,$$

is in X^{**}. In particular, for each E in Σ, $\nu(E) = \int_E f\,d\lambda$, defined to equal φ_{χ_E} and called the *Dunford integral* of f over E, is an element of X^{**}.

The function $\nu: \Sigma \to X^{**}$ is not necessarily countably additive. It can be shown that ν is countably additive if and only if T is a weakly compact operator if and only if $\{x^*f: \|x^*\| \leq 1\}$ is uniformly integrable in $L^1(\lambda)$ [**1**, pp. 319, 485, 292]. These conditions are automatically satisfied if f has bounded range.

Let $\hat{X}$ denote the natural image of X in X^{**}. The function f is said to be *Pettis integrable* if and only if for every E in Σ, $\nu(E)$ is in $\hat{X}$ (equivalently, $\nu(E)$ is weak* continuous on X^*). The following proposition is essentially a reformulation of the definition.

PROPOSITION 1. *A Dunford integrable function f is Pettis integrable if and only if the operator $T: X^* \to L^1(\lambda)$ is weak*-to-weak continuous.*

In particular, if f is Pettis integrable then T is necessarily a weakly compact operator.

PROOF. ($\Leftarrow$) is clear.

($\Rightarrow$) If f is Pettis integrable, then for each simple function g in $L^\infty(\lambda)$, φ_g is weak* continuous on X^*. By approximation, φ_g is weak* continuous for every g in $L^\infty(\lambda)$. $\square$

Therefore, to study Pettis integrability one studies the action of T on weak* neighborhoods in X^*. If F is a finite set in X, and $\varepsilon > 0$, let

$$\mathcal{K}(F, \varepsilon) = \{x^* \in X^*: \|x^*\| \leq 1 \text{ and } x^*(x) \leq \varepsilon \text{ for every } x \text{ in } F\}.$$

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LEMMA 2. *If f is Dunford integrable, then for all F, ε the set $T(\mathcal{K}(F,\varepsilon))$ is closed and convex in $L^1(\lambda)$.*

PROOF. Convexity is clear. Suppose g is in the closure of $T(\mathcal{K}(F,\varepsilon))$, and choose x_n^* in $\mathcal{K}(F,\varepsilon)$ with $x_n^* f \to g$ a.e. Let x^* be a weak* cluster point of $(x_n^*)_n$. Then x^* is in $\mathcal{K}(F,\varepsilon)$ and $g = x^* f$ a.e. □

The following reformulation of Proposition 1 was derived from ideas in proofs due to M. Talagrand (see Sentilles and Wheeler [**5**]).

PROPOSITION 3. *If f is Dunford integrable, then the following are equivalent:*
1. *f is Pettis integrable;*
2. *T is a weakly compact operator and*

$$\{0\} = \bigcap \{T(\mathcal{K}(F,\varepsilon)) | F \subset X, F \textit{ finite, and } \varepsilon > 0\}.$$

PROOF. (1) ⇒ (2) If f is Pettis integrable, then T is weakly compact. Suppose g is in $\bigcap_{(F,\varepsilon)} T(\mathcal{K}(F,\varepsilon))$. For each (F,ε) choose $x_{F,\varepsilon}^*$ in $\mathcal{K}(F,\varepsilon)$ so that $g = T(x_{F,\varepsilon}^*)$. Note that $(x_{F,\varepsilon}^*)_{(F,\varepsilon)}$ is naturally a net in X^* which converges weak* to 0. Hence, $g = T(x_{F,\varepsilon}^*) \to 0$.

(2) ⇒ (1) Let $B^* = \{x^* | \|x^*\| \leq 1\}$. Suppose a net (x_α^*) in $(1/2)B^*$ converges weak* to x^*. Then $(x_\alpha^* - x^*)$ is in B^* and for all (F,ε) it is eventually in $\mathcal{K}(F,\varepsilon)$. Let g be any weak cluster point of $(T(x_\alpha^* - x^*))$. Then g is in $\bigcap_{(F,\varepsilon)} T(\mathcal{K}(F,\varepsilon))$, so $g = 0$. Thus $T(x_\alpha^*) \to T(x^*)$ weakly in $L^1(\lambda)$. It follows that T is weak*-to-weak continuous. □

Say that a weakly measurable function $f: \Omega \to X$ is *separable-like* provided there exists a separable subspace D of X such that for every x^* in X^*,

$$x^* \chi_D f = x^* f \quad \text{a.e. } (\lambda).$$

(That is, as far as x^* is concerned, f takes almost all its range in D.) In particular, simple functions are separable-like. If (Ω, Σ, μ) is a separable measure space, then every Dunford integrable function is automatically separable-like.

COROLLARY 4. *Suppose f is Dunford integrable and T is weakly compact. If f is separable-like, then it is Pettis integrable.*

PROOF. Let (x_n) be dense in D. Let g be in $\bigcap_{(F,\varepsilon)} T(\mathcal{K}(F,\varepsilon))$. We must show that $g = 0$ a.e.

For each n, choose x_n^* in $\mathcal{K}(\{x_i\}_{i=1}^n, 1/n)$ so that $g = x_n^* f$ a.e. Now choose a fixed null set E so that for every n, $g = x_n^* f$ off E. Let $(x_n^*)_n$ cluster weak* at x^*. Then $g = x^* f$ off E, while $x^* = 0$ on D. Hence,

$$g = x^* f = x^* \chi_D f = 0 \quad \text{a.e.} \quad \square$$

If (Ω, Σ, μ) is a perfect measure space, Geitz [**3**] shows that every Pettis integrable f is separable-like. Thus, the converse of the Corollary holds for perfect measure spaces.

The next corollary is obvious.

COROLLARY 5. *Suppose f is Dunford integrable, T is weakly compact, and there is a sequence (f_n) of separable-like integrable functions such that for each x^*, $(x^* f_n)$ converges a.e. to $x^* f$. Then f is Pettis integrable.*

If $f:\Omega \to X$, then the *core* of f over a set E in Σ is defined to be the set

$$\operatorname{cor}_E f = \bigcap\{\overline{\mathrm{co}}(f(E\backslash N))|N \in \Sigma, \lambda(N) = 0\}.$$

LEMMA 6. *Suppose f is weakly measurable and that*

$$E \in \Sigma, \qquad \lambda(E) \neq 0 \Rightarrow \operatorname{cor}_E f \neq \varnothing.$$

If x^* is X^*, then $x^*f = 0$ a.e. on Ω if and only if x^* is constantly 0 on $\operatorname{cor}_\Omega f$.

PROOF. $(\Rightarrow)$ clearly.

$(\Leftarrow)$ If x^*f is not zero a.e., we may assume there exist E in Σ and $\alpha > 0$ such that $\lambda(E) \neq 0$ and $x^*f > \alpha$ on E. Then $\operatorname{cor}_E f \subset \{x|x^*(x) \geq \alpha\}$, and $\varnothing \neq \operatorname{cor}_E f \subset \operatorname{cor}_\Omega f$, so x^* is not constantly zero on $\operatorname{cor}_\Omega f$. □

COROLLARY 7 (GEITZ-TALAGRAND). *Suppose $f:\Omega \to X$ is Dunford integrable and T is weakly compact. Then f is Pettis integrable if and only if*

$$(*) \qquad E \in \Sigma, \qquad \lambda(E) \neq 0 \Rightarrow \operatorname{cor}_E f \neq \varnothing.$$

PROOF. $(\Rightarrow)$ If f is Pettis integrable, then by the separation theorem the integral $\int_E f\,d\lambda$ is in $\operatorname{cor}_E f$.

$(\Leftarrow)$ Suppose $(*)$ holds and g is in $\bigcap_{(F,\varepsilon)} T(\mathcal{K}(F,\varepsilon))$, with $g = x^*f$ for some x^* in X^*. If g is not identically zero a.e., then there exists x in $\operatorname{cor}_\Omega f$ with $x^*(x) \neq 0$.

For each n, choose x_n^* in $\mathcal{K}(\{x\}, 1/n)$ with $g = x_n^* f$ a.e. Choose a fixed null set E so that for every n, $g = x_n^* f$ off E. Let y^* be a weak* cluster point of (x_n^*). Then $y^*f = g$ a.e., and $y^*(x) = 0$.

Let $z^* = x^* - y^*$. Then $z^*f = 0$ a.e. while $z^*(x) \neq 0$, contradicting the lemma.

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