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ADDITIVITY OF JORDAN*-MAPS ON AW^*-ALGEBRAS

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ABSTRACT. Let M and N be AW^*-algebras and ϕ be a Jordan*-map from M to N which satisfies

(1) $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all x and y in M,

(2) $\phi(x^*) = \phi(x)^*$ for all $x \in M$, and

(3) ϕ is bijective, where $x \circ y = (1/2)(xy + yx)$.

If M has no abelian direct summand and a Jordan*-map ϕ is uniformly continuous on every abelian C^*-subalgebra of M, then we can conclude that ϕ is additive. Moreover, ϕ is the sum of ϕ_i $(i = 1, 2, 3, 4)$ such that ϕ_1 is a linear *-ring isomorphism, ϕ_2 is a linear *-ring anti-isomorphism, ϕ_3 is a conjugate linear *-ring anti-isomorphism and ϕ_4 is a conjugate linear *-ring isomorphism.

1. Preliminaries. The special Jordan product (resp. the special Jordan triple product) is defined by $x \circ y = (1/2)(xy + yx)$ (resp. $\{x, y, z\} = (1/2)(xyz + zyx)$).

DEFINITION 1.1. Let M and N be AW^*-algebras, and let ϕ be a map from M to N. If ϕ satisfies the following conditions (i)–(iii), then ϕ is called a Jordan*-map.

(i) $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for x and y of M,

(ii) $\phi(x^*) = \phi(x)^*$ for $x \in M$, and

(iii) ϕ is bijective.

Throughout this paper, we always assume that M and N are AW^*-algebras and ϕ is a Jordan*-map from M to N.

LEMMA 1.2. *Let e and f be projections. Then*

(i) *$ef = 0$ if and only if $e \circ f = 0$, and*

(ii) *$e = ef$ if and only if $e = e \circ f$.*

PROOF. (i) If $ef = 0$, then $e \circ f = 0$ is obvious. Suppose $e \circ f = 0$. Since $ef = -fe$, $ef = (ef)f = (-fe)f = (-fe)ef = (ef)^2 = e(fe)f = e(-ef)f = -ef$. So $ef = 0$. (ii) By the assertion (i), we have $e(1-f) = 0$ if and only if $e \circ (1-f) = 0$. Hence $e = ef$ if and only if $e = e \circ f$.

COROLLARY 1.3. *$\phi|M_p$ is a lattice isomorphism between the lattice M_p of projections of M and the lattice N_p of projections of N and preserves the orthogonality.*

COROLLARY 1.4. (i) $\phi(0) = 0$, $\phi(1) = 1$,

(ii) *If $\{e_i : i = 1, 2, \ldots, n\} \subset M_p$ is an orthogonal family, then $\phi(\sum_i \alpha_i e_i) = \sum_i \phi(\alpha_i e_i)$ for $\{\alpha_i : i = 1, 2, \ldots, n\} \subset \mathbf{C}$ (where $\mathbf{C}$ is the complex numbers), and*

(iii) *If $e \leq f$, then $\phi(f - e) = \phi(f) - \phi(e)$ (in particular $\phi(1 - e) = 1 - \phi(e)$).*

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PROOF. The assertions (i) and (iii) are obvious. If $\{e_i : i = 1, 2, \ldots, n\} \subset M_p$ is an orthogonal family, then $\{\phi(e_i) : i = 1, 2, \ldots, n\} \subset N_p$ is an orthogonal family. Put $x = \sum_i \alpha_i e_i$. Then

$$\begin{aligned}\phi(x) &= \phi(x) \circ \left(\bigvee_j e_j\right) = \phi(x) \circ \phi\left(\bigvee_j e_j\right) = \phi(x) \circ \left(\bigvee_j \phi(e_j)\right) \\ &= \phi(x) \circ \left(\sum_j \phi(e_j)\right) = \sum_j \phi(x \circ e_j) = \sum_j \phi(\alpha_j e_j).\end{aligned}$$

LEMMA 1.5. *The following assertions* (i) *and* (ii) *hold.*

(i) $\phi(-x) = -x$ *for every* $x \in M$.

(ii) *There exists a unique central projection* e_0 *of* M *such that* $\phi(i \cdot 1) = i\phi(e_0) - i\phi(1 - e_0)$ $(i^2 = -1)$.

PROOF. (i) Put $e_1 = \phi^{-1}((1/2)(1 + \phi(-1)))$. Then

$$\begin{aligned}\phi(-e_1) &= \phi(-1) \circ \phi(e_1) = \phi(-1) \circ ((1/2)(1 + \phi(-1))) \\ &= (1/2)(\phi(-1) + \phi(-1) \circ \phi(-1)) = \phi(e_1).\end{aligned}$$

Since ϕ is bijective, we get $-e_1 = e_1$ and so $e_1 = 0$. Therefore $\phi(-1) = -1$ and $\phi(-x) = \phi(-1) \circ \phi(x) = -\phi(x)$ follow for all $x \in M$.

(ii) Next, we shall show that $\phi(i \cdot 1)$ is a central element of N.

$$\begin{aligned}\phi(x) &= \phi(-((i \cdot 1) \circ x) \circ (i \cdot 1)) \\ &= -(1/2)\phi(i \cdot 1)\phi(x)\phi(i \cdot 1) + (1/2)\phi(x)\end{aligned}$$

for every $x \in M$. Hence $\phi(x) = -\phi(i \cdot 1)\phi(x)\phi(i \cdot 1)$ and so $\phi(i \cdot 1)\phi(x) = -\phi(i \cdot 1)^2\phi(x)\phi(i \cdot 1) = \phi(x)\phi(i \cdot 1)$. Therefore $\phi(i \cdot 1)$ is a central element of N.

Put $e_0 = \phi^{-1}((1/2)(1 - i\phi(i \cdot 1)))$. Then $\phi(e_0) = (1/2)(1 - i\phi(i \cdot 1))$ is a central projection of N. Since

$$\begin{aligned}\phi(e_0 \circ f) &= \phi(e_0) \circ \phi(f) = \phi(e_0)\phi(f) \\ &= \phi(e_0) \wedge \phi(f) = \phi(e_0 \wedge f)\end{aligned}$$

for all $f \in M_p$,

$$0 = e_0 \circ f - e_0 \wedge f = e_0 \circ (f - e_0 \wedge f) = e_0(f - e_0 \wedge f)$$

by Lemma 1.2 (i.e. $e_0 f = e_0 \wedge f \in M_p$). Hence, e_0 commutes with f, so e_0 is a central projection of M and

$$\phi(i \cdot 1) = i\phi(e_0) - i(1 - \phi(e_0)) = i\phi(e_0) - i\phi(1 - e_0).$$

Finally, we shall show that e_0 is unique. Suppose $\phi(i \cdot 1) = if - i(1 - f)$ for some projection f in N. Then $if = \phi(i \cdot 1)f = i\phi(e_0)f - i\phi(1 - e_0)f$.

Since $if - i\phi(e_0)f = -i(1 - \phi(e_0))f$, we get $(1 - \phi(e_0))f = 0$ and so $f \le \phi(e_0)$. Therefore $f = \phi(e_0)$ by the symmetry of $\phi(e_0)$ and f.

LEMMA 1.6. *$\phi(exe) = \phi(e)\phi(x)\phi(e)$ holds for any projection e in M and any x in M.*

PROOF. Since $\phi(2e-1) = \phi(e) - \phi(1-e) = 2\phi(e) - 1$,

$$\begin{aligned}\phi(exe) &= \phi(((2e-1)\circ x)\circ e) = (\phi(2e-1)\circ\phi(x))\circ\phi(e)\\ &= ((2\phi(e)-1)\circ\phi(x))\circ\phi(e) = \phi(e)\phi(x)\phi(e).\end{aligned}$$

2. AW^*-algebra which has an $n\times n$ $(n\geq 2)$ matrix unit. Throughout this paper, we suppose that M has an $n\times n$ $(n\geq 2)$ matrix unit.

LEMMA 2.1. (i) *$\phi|\mathbf{C}\cdot 1$ is additive.* (ii) *For every $x\in M$, $\phi(\rho x) = \rho\phi(x)$ holds for all rational ρ.*

PROOF. Let $\{v_{ij}\}$ be the matrix unit of M. Put $e = v_{ii}$, $v = v_{ij}$ $(i\neq j)$, $p = (1/2)(e+v^*)(e+v)$ and $q = (1/2)(e-v^*)(e-v)$. Since p and q are orthogonal projections in M, we have

$$\begin{aligned}\phi((\alpha+\beta)e) &= \phi(e(2\alpha p + 2\beta q)e) = \phi(e)\phi(2\alpha p + 2\beta q)\phi(e)\\ &= \phi(e)(\phi(2\alpha p) + \phi(2\beta q))\phi(e) = \phi(\alpha e) + \phi(\beta e)\end{aligned}$$

by Lemma 1.6 and Corollary 1.4. So our assertion (i) follows. Let n be an arbitrary integer and m be a natural number. Then, for every $x\in M$,

$$\begin{aligned}n\phi(x) &= \phi(n\cdot 1)\circ\phi(x) = \phi(nx) = \phi(m((n/m)x))\\ &= \phi(m\cdot 1)\circ\phi((n/m)x) = m\phi((n/m)x)\end{aligned}$$

follows. So we have assertion (ii).

COROLLARY 2.2. *Let e and f be orthogonal projections of M. Then $\phi(\{e,x,f\}) = \{\phi(e),\phi(x),\phi(f)\}$ holds.*

PROOF. Since $2(e\circ x)\circ f = \{e,x,f\}$ and $\phi(e)\phi(f) = 0$ (Lemma 1.2), we have

$$\begin{aligned}\phi(\{e,x,f\}) &= \phi(2(e\circ x)\circ f) = 2(\phi(e)\circ\phi(x))\circ\phi(f)\\ &= \{\phi(e),\phi(x),\phi(f)\}.\end{aligned}$$

LEMMA 2.3. *$\phi(\lambda\cdot 1) = \lambda\cdot 1$ holds for all $\lambda\in\mathbf{R}$ (where $\mathbf{R}$ is the real numbers).*

PROOF. Since $\phi|\mathbf{C}\cdot 1$ is additive, $\phi(\rho\cdot 1) = \rho\cdot 1$ for every rational number ρ. Let $[\lambda]$ be the integral part of $\lambda\in\mathbf{R}$. Then we have

$$0\leq\phi(\lambda\cdot 1)\leq\phi([1/\lambda]^{-1}\cdot 1)\leq[1/\lambda]^{-1}\cdot 1\quad\text{for all }\lambda\in(0,1).$$

Since $\phi(-1) = -1$, the map $\lambda\mapsto\phi(\lambda\cdot 1)$ is continuous at 0. Hence, the map is continuous on $\mathbf{R}$.

Therefore we get $\phi(\lambda\cdot 1) = \lambda\cdot 1$ for all $\lambda\in\mathbf{R}$.

LEMMA 2.4. *There exists a unique central projection e_0 of M such that $\phi(\alpha\cdot 1) = \alpha\phi(e_0) + \bar{\alpha}\phi(1-e_0)$ for any $\alpha\in\mathbf{C}$.*

PROOF. For every $\lambda,\mu\in\mathbf{R}$,

$$\begin{aligned}\phi((\lambda+i\mu)\cdot 1) &= \phi(\lambda\cdot 1) + \phi(i\cdot 1)\phi(\mu\cdot 1)\\ &= \lambda\cdot 1 + (i\phi(e_0) - i\phi(1-e_0))(\mu\cdot 1)\\ &= (\lambda+i\mu)\phi(e_0) + (\lambda-i\mu)\phi(1-e_0)\qquad (i^2=-1)\end{aligned}$$

by Lemma 2.1 and Lemma 1.5.

LEMMA 2.5. *Let $a = \sum_i \alpha_i e_i$ where α_i $(i = 1, 2, \ldots, n)$ are in $\mathbf{C}$ and e_i $(i = 1, 2, \ldots, n)$ are orthogonal projections such that $\sum_i e_i = 1$. Then*

$$\phi(axa) = \phi(a)\phi(x)\phi(a) \quad \textit{for all } x \in M.$$

PROOF. Since $\sum_i \phi(e_i) = 1$ (Corollary 1.4),

$$\begin{aligned}
\phi(axa) &= \left(\sum_i \phi(e_i)\right)\phi(axa)\left(\sum_i \phi(e_i)\right) \\
&= \sum_i \phi(e_i)\phi(axa)\phi(e_i) + 2\sum_{i<j}\{\phi(e_i), \phi(axa), \phi(e_j)\} \\
&= \sum_i \phi(e_i axa e_i) + 2\sum_{i<j}\phi(\{e_i, axa, e_j\}) \\
&= \sum_i \phi(\alpha_i \cdot 1)^2 \phi(e_i)\phi(x)\phi(e_i) \\
&\quad + 2\sum_{i<j}\phi(\alpha_i \cdot 1)\phi(\alpha_j \cdot 1)\{\phi(e_i), \phi(x), \phi(e_j)\} \\
&= \phi(a)\phi(x)\phi(a)
\end{aligned}$$

by Lemma 1.6 and Corollary 2.2.

3. Structure of Jordan*-maps. In this section we assume that ϕ satisfies the following condition:

(iv) ϕ is uniformly continuous on every abelian C^*-algebra of M.

LEMMA 3.1. *If $h \in M$ is selfadjoint, then $\phi|C^*(h, 1)$ is additive where $C^*(h, 1)$ is the C^*-subalgebra which is generated by h and 1.*

PROOF. Let $h = \int_{\sigma(h)} \lambda \, de_\lambda$ be the spectral decomposition of h, where $\sigma(h)$ is the spectrum of h.

For any x and y in $C^*(h, 1)$, there exist f and g in $\mathcal{C}(\mathbf{R})$ ($\mathcal{C}(\mathbf{R})$ is the C^*-algebra of the complex-valued continuous functions on $\mathbf{R}$) such that

$$x = \int_{\sigma(h)} f(\lambda) de_\lambda = \lim \sum_j f(\lambda_j) e_j$$

and

$$y = \int_{\sigma(h)} g(\lambda) de_\lambda = \lim \sum_j g(\lambda_j) e_j.$$

So we have

$$\begin{aligned}
\phi(x + y) &= \lim \sum_j \phi((f(\lambda_j) + g(\lambda_j)) \cdot 1)\phi(e_j) \\
&= \lim \sum_j (\phi(f(\lambda_j) \cdot 1) + \phi(g(\lambda_j) \cdot 1))\phi(e_j) \\
&= \phi(x) + \phi(y)
\end{aligned}$$

by the condition (iv), Corollary 1.4(ii) and Lemma 2.1.

LEMMA 3.2. *Let u and v be unitaries in M. If u is selfadjoint, then we have* $\phi(u+v) = \phi(u) + \phi(v)$.

PROOF. Put $e = (1/2)(1+u)$ and $w = e + i(1-e)$ $(i^2 = -1)$. Then e is a projection in M and w is a unitary in M such that $w^2 = u$. Since w^*vw^* is a unitary in M, by the spectral theory, there exists a selfadjoint element h in M such that $w^*vw^* = e^{ih}$.

The map $f \in \mathcal{C}(\sigma(h)) = \mathcal{C}(\mathbf{R})|\sigma(h) \mapsto f(h) \in C^*(h, 1)$ is a surjective isometric *-isomorphism from $\mathcal{C}(\sigma(h))$ to $C^*(h,1)$. So

$$\left\| e^{ih} - \sum_{k=0}^{n}((ih)^k/k!) \right\| = \sup\left\{ \left| e^{i\lambda} - \sum_{k=0}^{n}((i\lambda)^k/k!) \right| : \lambda \in \sigma(h) \right\} \to 0 \quad (\text{as } n \to \infty).$$

Hence $w^*vw^* \in C^*(h,1) \subset M$.

Thus it follows that $\phi(1 + w^*vw^*) = \phi(1) + \phi(w^*vw^*)$ by Lemma 3.1. So we get

$$\begin{aligned} \phi(u+v) &= \phi(w(1+w^*vw^*)w) = \phi(w)\phi(1+w^*vw^*)\phi(w) \\ &= \phi(w)(\phi(1) + \phi(w^*vw^*))\phi(w) \\ &= \phi(u) + \phi(v) \quad \text{by Lemma 2.5.} \end{aligned}$$

For some pair of projections e and f in M, we write $e \sim f$ (resp. $e \precsim f$) if there exists a partial isometry v in M such that $vv^* = e$ and $v^*v = f$ (resp. $v^*v \le f$).

LEMMA 3.3. *Let h be a nonzero selfadjoint element in M, x be a nonzero element in M and e be a projection in M such that $e \precsim 1-e$. Then*

$$\phi(ehe + exe) = \phi(ehe) + \phi(exe).$$

PROOF. First of all, we shall note that for any $x \in M$ with $\|x\| \le 1$, there exists a unitary u such that $exe = eue$. In particular, when x is selfadjoint, u also is selfadjoint. In fact, let v be a partial isometry in M such that $vv^* = e$ and $v^*v \le 1-e$. If we put u to be

$$y + (e - yy^*)^{(1/2)}v + v^*(e - y^*y)^{(1/2)} - vy^*v^* + g$$

where $y = exe$ and $g = 1 - (e + v^*v)$, u satisfies all the requirements [**2**].

If we put $\gamma(x,y) = \|x\| + \|y\|$, $h_1 = \gamma(h,x)^{-1}h$ and $x_1 = \gamma(h,x)^{-1}x$, then there exist unitaries u and v in M such that $eh_1e = eue$, $ex_1e = eve$ and u is selfadjoint. And it follows that

$$\begin{aligned} \phi(ehe + exe) &= \gamma(h,x)\phi(e(u+v)e) = \gamma(h,x)\phi(e)\phi(u+v)\phi(e) \\ &= \gamma(h,x)\phi(e)(\phi(u) + \phi(v))\phi(e) = \phi(ehe) + \phi(exe) \end{aligned}$$

by Lemmas 1.6 and 3.2.

LEMMA 3.4. *Suppose e is a projection in M such that $e \precsim 1-e$. Then $\phi|eMe$ is additive.*

PROOF. Take arbitrary x and y in M and put $x = h + ik$ $(i^2 = -1)$ where h and k are selfadjoint.

Suppose h, k and y are nonzero. Then

$$\begin{aligned}\phi(exe + eye) &= \phi(ehe) + \phi(e(ik + y)e) \\ &= \phi(ehe) + \phi(i \cdot 1)(\phi(eke) + \phi(-ieye)) \\ &= \phi(ehe + e(ik)e) + \phi(eye) \\ &= \phi(exe) + \phi(eye)\end{aligned}$$

by Lemma 3.3. When $h = 0$, $k = 0$ or $y = 0$, the above equalities also hold.

LEMMA 3.5. *Suppose e and f are projections in M such that $e \sim f \leq 1 - e$. Then for any selfadjoint element h in M with $\|h\| \leq 1$, there exists a selfadjoint unitary u in M such that $\{e, h, f\} = \{e, u, f\}$.*

PROOF. Put u to be

$$u = a + a^* + (e - aa^*)^{(1/2)} - (f - a^*a)^{(1/2)} + g$$

where $a = ehf$ and $g = 1 - (e + f)$. Then u satisfies all the requirements.

LEMMA 3.6. *Let h and k be selfadjoint elements in M with $\|h\| \leq 1$ and $\|k\| \leq 1$ and let $\alpha \in \mathbf{C}$ with $|\alpha| = 1$. Suppose e and f are orthogonal equivalent projections in M, then we have*

$$\phi(\{e, h, f\} + \alpha\{e, k, f\}) = \phi(\{e, h, f\}) + \phi(\alpha\{e, k, f\}).$$

PROOF. Put $h_1 = \gamma(h, k)^{-1}h$ and $k_1 = \gamma(h, k)^{-1}k$. Then there exist selfadjoint unitaries u and v such that $\{e, h_1, f\} = \{e, u, f\}$ and $\{e, k_1, f\} = \{e, v, f\}$, and it follows that

$$\begin{aligned}\phi(\{e, h, f\} + \alpha\{e, k, f\}) &= \gamma(h, k)\phi(\{e, u + \alpha v, f\}) \\ &= \gamma(h, k)\{\phi(e), \phi(u + \alpha v), \phi(f)\} \\ &= \gamma(h, k)\{\phi(e), \phi(u) + \phi(\alpha v), \phi(f)\} \\ &= \phi(\{e, h, f\}) + \phi(\alpha\{e, k, f\})\end{aligned}$$

by Lemmas 3.5 and 3.2.

Put $x = h_1 + ik_1$, $y = h_2 + ik_2$ ($i^2 = -1$) where h_j, k_j ($j = 1, 2$) are selfadjoint elements in M. Then Lemma 3.6 leads to the following

COROLLARY 3.7. *Let e and f be orthogonal equivalent projections in M. Then $\phi|\{e, M, f\}$ is additive where $\{e, M, f\} = \{\{e, x, f\}: x \in M\}$.*

THEOREM 3.8. *Let M and N be AW^*-algebras and let ϕ be a Jordan*-map from M to N. Suppose that M has no abelian direct summand and ϕ is uniformly continuous on each abelian C^*-subalgebra of M. Then ϕ is additive.*

PROOF. Let $\{p_i\}$ be a family of central orthogonal projections such that $\bigvee_i p_i = 1$ where Mp_1 has no finite type I direct summand and Mp_i ($i \geq 2$) is homogeneous type I_{n_i} for some natural number n_i. Then $\phi|Mp_i$ is a Jordan*-map from Mp_i to $N\phi(p_i)$. We can identify x with $\bigoplus_i xp_i$ (C^*-sum) and $\phi(x)$ with $\bigoplus_i \phi(x)\phi(p_i)$. Therefore it is sufficient to prove about Mp_i for every p_i, and we may assume that

M has an $n \times n$ $(n \geq 2)$ matrix unit. Let $\{e_i : i = 1, 2, \ldots, n\}$ be the family of diagonal projections of the matrix unit of M. Since $\sum_i \phi(e_i) = 1$, we have

$$\begin{aligned} \phi(x) &= \left(\sum_i \phi(e_i)\right) \phi(x) \left(\sum_i \phi(e_i)\right) \\ &= \sum_i \phi(e_i)\phi(x)\phi(e_i) + 2\sum_{i<j} \{\phi(e_i), \phi(x), \phi(e_j)\} \\ &= \sum_i \phi(e_i x e_i) + 2\sum_{i<j} \phi(\{e_i, x, e_j\}). \end{aligned}$$

Since $\phi|e_iMe_i$ and $\phi|\{e_i, M, e_j\}$ are additive, ϕ is additive.

The next lemma is due to R. V. Kadison [**4**]. He proved it in the case of von Neumann algebras. However, his proof holds in the case of AW^*-algebras with a slight modification of terminologies.

LEMMA 3.9 [4, THEOREM 10]. *Let M (resp. N) be an AW^*-algebra (resp. C^*-algebra) and let ϕ be a C^*-isomorphism from M to N. Then there exists a central projection f_0 in M such that $\phi|Mf_0$ (resp. $\phi|M(1-f_0)$) is a *-ring isomorphism (resp. *-ring anti-isomorphism).*

THEOREM 3.10. *Keep the assumptions on M, N and ϕ as in Theorem 3.8. There exist four central projections e_1, e_2, e_3, e_4 in M such that $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \phi_4$ where $\phi_i = \phi|Me_i$ $(i = 1, 2, 3, 4)$. Then ϕ_1 is a linear *-ring isomorphism, ϕ_2 is a linear *-ring anti-isomorphism, ϕ_3 is a conjugate linear *-ring isomorphism and ϕ_4 is a conjugate linear anti-isomorphism.*

PROOF. By Theorem 3.8 and Lemma 2.4, there exists a unique central projection e_0 in M such that $\phi|Me_0$ is a C^*-isomorphism of Me_0 onto $N\phi(e_0)$ and $\phi|M(1-e_0)$ is a conjugate linear map from $M(1-e_0)$ onto $N\phi(1-e_0)$ which preserves *-operation and special Jordan product. Put $\psi(x) = \phi(xe_0) + (\phi(x(1-e_0)))^*$. Then ψ is a C^*-isomorphism between M and N. So there exists a central projection f_0 in M such that $\psi|Mf_0$ (resp. $\psi|M(1-f_0)$) is a linear *-ring isomorphism (resp. linear *-ring anti-isomorphism).

Therefore, we put $e_1 = e_0 f_0$, $e_2 = e_0(1-f_0)$, $e_3 = (1-e_0)(1-f_0)$ and $e_4 = (1-e_0)f_0$; then e_1, e_2, e_3 and e_4 satisfy all the requirements.

REMARK. There is an example where the projections e_1, e_2, e_3 and e_4 in Theorem 3.10 are all nontrivial. In fact, let $M = N = B(H_2) \oplus B(H_2) \oplus B(H_2) \oplus B(H_2)$ where H_2 is the 2-dimensional Hilbert space and $x = (x_{ij}) \in B(H_2)$. Let $\phi_1(x) = x$, $\phi_2(x) = {}^t(x_{ij})$ (transpose of x), $\phi_3(x) = (\overline{x_{ij}})$, $\phi(x) = x^*$ and $\phi = \bigoplus_{j=1}^4 \phi_j$. Put $e_i = \bigoplus_{j=1}^4 \delta_{ij} \cdot 1$ $(i = 1, 2, 3, 4)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if not (Kronecker's δ). Then ϕ is a Jordan*-map from M to M, and all e_i $(i = 1, 2, 3, 4)$ are nontrivial, satisfying the requirements of Theorem 3.10.

4. Conjectures.

CONJECTURE 4.1 (S. SAKAI). Theorems 3.8 and 3.10 hold without any hypothesis of continuity.

CONJECTURE 4.2 (K. SAITÔ). Versions of those theorems hold among JBW-algebras.

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