## A SINGULAR INTEGRAL

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ABSTRACT. In this paper we show that if  $K(x) = \Omega(x)/|x|^n$  is a Calderón-Zygmund kernel, where  $\Omega \in L^q(S^{n-1})$  for some  $1 < q \le \infty$ , and b is a radial bounded function, then b(x)K(x) is the kernel of a convolution operator which is bounded on  $L^p(\mathbb{R}^n)$  for  $1 and <math>n \ge 2$ .

This paper is related to boundedness properties of a variation of Calderón-Zygmund operators.

Let  $H(x) = b(|x|)\Omega(x)/|x|^n$  be a kernel, where  $\int_{S^{n-1}} \Omega(x) d\sigma(x') = 0$ ,  $\Omega(\lambda x) = \Omega(x)$  for  $x \in \mathbb{R}^n$ ,  $\lambda > 0$ , and b is radial. Define

(1) 
$$f(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} H(y) f(x-y) \, dy = \text{P.V. } H * f(x).$$

If  $b \equiv 1$  and  $\Omega \in L^q(S^{n-1})$  for some q > 1, then T is a Calderón-Zygmund operator which is bounded on  $L^p(\mathbb{R}^n)$  for  $1 and <math>n \ge 1$ ; see [4].

In [1] R. Fefferman showed that if  $\Omega$  satisfies a Lipschitz condition and b is bounded, then T is bounded on  $L^p(\mathbb{R}^n)$ ,  $1 , <math>n \ge 2$ . This is not true when n = 1; for instance, if  $H(x) = \sin(|x|)/x$ , then its Fourier transform  $\hat{H}(\xi)$  is unbounded, indicating that T cannot be bounded on  $L^2(\mathbb{R}^1)$ . It turns out that the smoothness condition on  $\Omega$  can be relaxed a great deal.

THEOREM. If  $\Omega \in L^q(S^{n-1})$  for some  $1 < q \le \infty$ , and  $b(|x|) \in L^{\infty}$ , then T is bounded on  $L^p(\mathbb{R}^n)$  for  $1 and <math>n \ge 2$ .

PROOF. First we show that the Fourier transform of H is bounded; thereby T is bounded on  $L^2(\mathbb{R}^n)$ . Using polar coordinates and letting  $x = \rho x'$ ,  $\rho = |x|$ , we get

(2) 
$$\hat{H}(\xi) = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{b(|x|)}{|x|^n} \Omega(x) e^{-ix \cdot \xi} dx$$
$$= \lim_{\varepsilon \to 0} \int_{\varepsilon < \rho < 1/\varepsilon} \frac{b(\rho)}{\rho} d\rho \left( \int_{S^{n-1}} \Omega(x') e^{i\rho x' \cdot \xi} d\sigma(x') \right).$$

By a change of variable  $r = \rho |\xi|$ , (2) becomes

(3) 
$$\hat{H}(\xi) = \lim_{\varepsilon \to 0} \int_{\varepsilon|\xi|}^{|\xi|/\varepsilon} \frac{b(r/|\xi|)}{r} dr \left( \int_{S^{n-1}} \Omega(x') e^{-irx' \cdot \xi'} d\sigma(x') \right)$$
$$= \lim_{\varepsilon \to 0} \int_{\varepsilon|\xi|}^{|\xi|/\varepsilon} \frac{b(r/|\xi|)}{r} g(r,\xi') dr$$

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where

$$g(r,\xi') = \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'} \, d\sigma(x'), \qquad |\xi'| = 1.$$

Now if

(4) 
$$\sup_{|\xi'|=1} \|g(\cdot,\xi')\|_{L^p(1,\infty)} \le A$$

for some  $1 and a constant A, then we claim that <math>\hat{H}(\xi)$  is bounded. To prove the claim, first we observe that

$$|g(r,\xi')| = \left| \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'} d\sigma(x') \right|$$
$$= \left| \int_{S^{n-1}} \Omega(x') (e^{-irx'\cdot\xi'} - 1) d\sigma(x') \right|$$
$$\leq 2r \|\Omega\|_{L^1(S^{n-1})} = cr$$

by the mean value theorem. Using this and Hölder's inequality, we see that

$$\begin{split} \int_0^\infty \left| \frac{b(r/|\xi|)}{r} g(r,\xi') \right| \, dr &\leq \|b\|_\infty \int_0^\infty \frac{|g(r,\xi')|}{r} \, dr \\ &\leq \|b\|_\infty \left( \int_0^1 \frac{cr}{r} \, dr + \|g(\cdot,\xi')\|_{L^p(1,\infty)} \cdot \left\| \frac{1}{r} \right\|_{L^{p'}(1,\infty)} \right) \\ &\leq \|b\|_\infty (c+A\|1/r\|_{L^{p'}(1,\infty)}) \leq c, \end{split}$$

where 1/p' = 1 - 1/p. Hence, by Lebesgue's dominated convergence theorem, the limit in (3) exists and  $|\hat{H}(\xi)| \leq c$ .

Now let us see when condition (4) may hold. Let  $\rho$  be a rotation such that  $\rho(\xi') = (1, 0, 0, \dots, 0)$ . In the integral of  $g(r, \xi')$  make the change of variable  $x' \to \rho^{-1}x'$ , and let  $\Omega^{\rho}(x') = \Omega(\rho^{-1}x')$ . Then

$$g(r,\xi') = \int_{S^{n-1}} \Omega(x') e^{-irx'\cdot\xi'} \, d\sigma(x') = \int_{S^{n-1}} \Omega^{\rho}(x') e^{-ir(\rho^{-1}x')\cdot\xi'} \, d\sigma(x')$$
$$= \int_{S^{n-1}} \Omega^{\rho}(x') e^{-irx'\cdot\rho\xi'} = \int_{S^{n-1}} \Omega^{\rho}(x') e^{-irx'_1} \, d\sigma(x'),$$

where  $x' = (x'_1, x'_2, \dots, x'_n) = (x'_1, x'')$ . Taking  $x'_1 = \cos \theta$ , we can write the last integral as

$$g(r,\xi') = \int_0^{\pi} e^{-ir\cos\theta} d\theta \left( \int_{S_{\cos\theta}} \Omega^{\rho}(\cos\theta, x'') d\sigma_{\theta}(x') \right),$$

where  $S_{\cos\theta} = \{|x'| = 1 : x' = (\cos\theta, x'')\}$  and  $d\sigma_{\theta}(x')$  is its surface measure. By a further change of variable  $s = \cos\theta$ , the last integral is reduced to

$$g(r,\xi') = \int_{-1}^{1} e^{-irs} (1-s^2)^{-1/2} \left( \int_{S_s} \Omega^{\rho}(s,x'') \, d\sigma_s(x') \right) \, ds.$$

Set  $x'' = \sqrt{1 - s^2}y'$ . Then  $y' \in S^{n-2}$  and

(5) 
$$g(r,\xi') = \int_{-1}^{1} e^{-irs} (1-s^2)^{(n-3)/2} \left( \int_{S^{n-2}} \Omega^{\rho}(s,\sqrt{1-s^2}y') \, d\sigma_{n-2}(y') \right),$$

where  $d\sigma_{n-2}(y')$  is the surface measure of  $S^{n-2}$ .

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From (5) we observe that  $g(r, \xi') = (h(s)w(s))^{\hat{}}(r)$ , where

$$h(s) = \int_{S^{n-2}} \Omega^{\rho}(s, \sqrt{1-s^2}y') \, d\sigma_{n-2}(y') \quad \text{and} \quad w(s) = (1-s^2)^{(n-3)/2} \chi_{(-1,1)}(s).$$

Without loss of generality, we may assume that  $\Omega \in L^q(S^{n-1})$  for some  $1 < q < \infty$ . Let q' be the conjugate of q; 1/q' + 1/q = 1. By Hölder's inequality,

$$\begin{aligned} |h(s)| &= \left| \int_{S^{n-2}} \Omega^{\rho}(s, \sqrt{1-s^2}y') \, d\sigma_{n-2}(y') \right| \\ &\leq \omega_{n-2}^{1/q'} \left( \int_{S^{n-2}} |\Omega^{\rho}(s, \sqrt{1-s^2}y')|^q \, d\sigma_{n-2}(y') \right)^{1/q} \\ &= c \left( \int |\Omega^{\rho}(s, \sqrt{1-s^2}y')|^q \, d\sigma_{n-2}(y') \right)^{1/q}. \end{aligned}$$

 $\omega_{n-2}$  is the area of  $S^{n-2}$ . Hence,

$$\begin{split} \int_{-1}^{1} |h(s)|^{q} w(s) \, ds &\leq c \int_{-1}^{1} (1-s^{2})^{(n-3)/2} \, ds \int_{S^{n-2}} |\Omega^{\rho}(s,\sqrt{1-s^{2}}y')|^{q} \, d\sigma_{n-2}(y') \\ &= c \|\Omega\|_{L^{q}(S^{n-1})}^{q}. \end{split}$$

Therefore,  $|h(s)|^q w(s) \in L^1(-1, 1)$ .

Let 1 < p' < q and r = q/p'. Using Hölder's inequality again,

$$\begin{split} \int_{-1}^{1} |h(s)w(s)|^{p'} ds &= \int_{-1}^{1} |h(s)|^{p'} w(s)^{1/r} w(s)^{p'-1/r} ds \\ &\leq \left( \int_{-1}^{1} |h(s)|^{q} w(s) ds \right)^{1/r} \left( \int_{-1}^{1} w(s)^{r'(p'-1/r)} ds \right)^{1/r'} \\ &\leq c \|\Omega\|_{L^{q}(S^{n-1})}^{q/r} \left( \int_{-1}^{1} (1-s^{2})^{[(n-3)/2]r'(p'-1/r)} ds \right)^{1/r'} \\ &= c \|\Omega\|_{L^{q}(S^{n-1})}^{p'} \left( \int_{-\pi/2}^{\pi/2} |\sin \theta|^{(n-3)r'(p'-1/r)+1} d\theta \right)^{1/r'}. \end{split}$$

This last integral is convergent if (n-3)r'(p'-1/r)+1 > -1. This inequality, obviously, holds for all p' when  $n \ge 3$ . In the case n = 2, it holds whenever 1 < p' < 2/(1+1/q). Therefore, if  $\Omega \in L^q(S^{n-1})$ ,  $1 < q \le \infty$ , then there always exists some p', 1 < p' < 2, such that  $h(s)w(s) \in L^{p'}(-1,1)$ . By the Hausdorff-Young inequality

$$\begin{split} \|g(\cdot,\xi')\|_{L^p(1,\infty)} &\leq \|g(\cdot,\xi')\|_{L^p(-\infty,\infty)} = \|(h(s)w(s))^{\widehat{}}\|_{L^p(-\infty,\infty)} \\ &\leq \|h(s)w(s)\|_{L^{p'}(-\infty,\infty)} \\ &= \|h(s)w(s)\|_{L^{p'}(-1,1)} \leq c \|\Omega\|_{L^q(S^{n-1})}^{p'} = A. \end{split}$$

This proves that  $\hat{H}(\xi)$  is bounded, and thus T is bounded on  $L^2(\mathbb{R}^n)$ ,  $n \ge 2$ .

Using the technique of complex interpolation and an argument similar to the one in [1], the  $L^p$  boundedness can be established. We mention it briefly.

Define

$$m_z(\xi) = \text{P.V.} \int_{\mathbb{R}^n} \frac{H(x)}{|x|^z} e^{-ix \cdot \xi} \, dx \cdot |\xi|^{-z}$$

and

$$(T_z f)\widehat{}(\xi) = m_z(\xi)\widehat{f}(\xi),$$

 $|\operatorname{Re}(z)| < \eta$  for some  $\eta > 0$ .  $T_z$  is an analytic family of operators in the sense of Stein. Also  $T_0 = T$ . Each  $T_z$  is a principal-valued convolution operator,  $T_z f = P.V.H_z * f$ , where  $H_z = m_z$ . As proved in [1], if  $0 < \operatorname{Re}(z) \leq \eta$  and  $\eta$  small enough, then

(6) 
$$\sup_{|y|>0}\int_{|x|>2|y|}|H_{z}(x+y)-H_{z}(x)|\,dx\leq c_{z},$$

where the constant  $c_z$  depends only on the real part of z. Also a similar argument as in the  $L^2$  case will show that  $|m_z(\xi)| \leq c$  for  $|\operatorname{Re}(z)| \leq \eta$ , when  $\eta$  is small enough. Therefore  $||T_{-\eta+iy}||_2 \leq c$  and  $||T_{\eta+iy}||_r \leq c_\eta$  for  $-\infty < y < \infty$  and  $1 < r < \infty$ . By complex interpolation, if 1/p = (1 - 1/2)/2 + (1/2)/r then  $||T||_p = ||T_0||_p \leq c$ . Hence, T is bounded on  $L^p(\mathbb{R}^n)$ ,  $1 , <math>n \geq 2$ .

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